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## Steady State Transmission Line Analysis

### A. Line Equations

Returning to our transmission line model complete with loss elements, we will now consider a very important special case, namely steady state as excitation. The model, redrawn in Fig. 1, is excited from the left with an ac voltage source at frequency $\omega$.

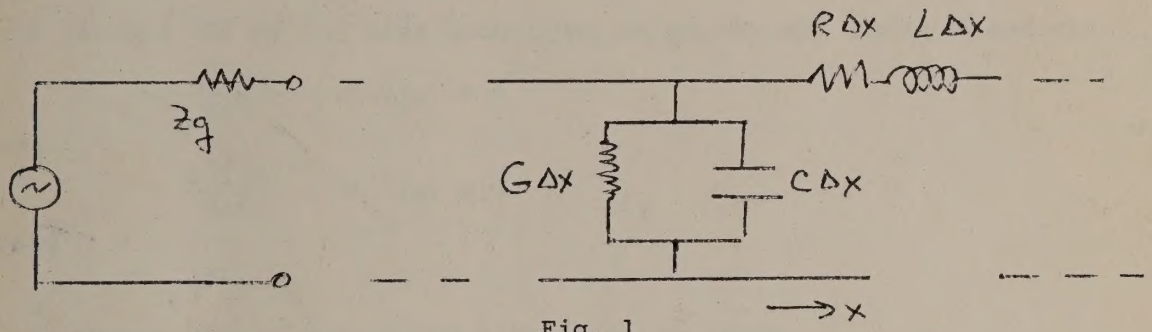



Fig. 1

Since all voltages and currents now vary harmonically with time, the actual voltages and currents are found from the rms phasor quantities from the prescription

$$\begin{aligned} v(x,t) &= \sqrt{2}\ \mathrm{Re}\ \bar{V}(x)\ e^{j\omega t} \\ i(x,t) &= \sqrt{2}\ \mathrm{Re}\ \bar{I}(x)\ e^{j\omega t} \end{aligned} \tag{1}$$

where

$$\begin{aligned} \bar{V}(x) &= |\bar{V}(x)|\ e^{j\theta_V} \\ \bar{I}(x) &= |\bar{I}(x)|\ e^{j\theta_I} \quad . \end{aligned}$$

The differential equations of the line are obtained as in the previous case. Now each equation includes an additional term. First, the series resistive drop across R adds to the inductive voltage drop resulting in

$$\frac{\partial v}{\partial x} = -Ri - L\frac{\partial i}{\partial t} \quad . \tag{2}$$

In the second equation the shunt current through G adds to that through C yielding

$$\frac{\partial i}{\partial x} = -Gv - C\frac{\partial v}{\partial t} \quad . \tag{3}$$

The time derivatives in Eqs. 2 and 3 can now be calculated since the time dependence is known a priori. For this case, using (1), equations 2 and 3 become

$$\begin{aligned} \frac{d\bar{V}(x)}{dx} &= -R\ \bar{I}(x) - j\omega L\ \bar{I}(x) \\ &= -Z\ \bar{I}(x) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{d\bar{I}(x)}{dx} &= -G\ \bar{V}(x) - j\omega C\ \bar{V}(x) \\ &= -Y\ \bar{V}(x) \quad . \end{aligned} \tag{5}$$

Here

$$Z = R + j\omega L \tag{6}$$

which represents the series impedance, and

$$Y = G + j\omega C \tag{7}$$

which represents the shunt admittance. One should note carefully that Z and Y so defined are not related, that is Z is not equal to 1/Y.

Differential equations for $\bar{V}(x)$ alone and $\bar{I}(x)$ alone can be obtained from (4) and (5) by the same techniques as previously employed and are

$$\frac{d^2\bar{V}(x)}{dx^2} - \gamma^2 \bar{V}(x) = 0 \tag{8}$$

and

$$\frac{d^2\bar{I}(x)}{dx^2} - \gamma^2 \bar{I}(x) = 0 \tag{9}$$

where

$$\gamma^2 = ZY \quad . \tag{10}$$

Solutions to Eqs. 8 and 9 exist in the form

$$\bar{V}(x) = \bar{V}^+ e^{-\gamma x} + \bar{V}^- e^{+\gamma x} \tag{11}$$

$$\bar{I}(x) = \bar{I}^+ e^{-\gamma x} + \bar{I}^- e^{+\gamma x} \tag{12}$$

where the $\bar{V}$ and $\bar{I}$ are complex phasors. Here again $\bar{V}^+$ and $\bar{I}^+$ are related through Eqs. 4 or 5. For example, using Eq. 4

$$\frac{d\bar{V}(x)}{dx} = -\bar{V}^+ e^{-\gamma x} = -Z \bar{I}^+ e^{-\gamma x}$$

then, using (10)

$$\frac{\bar{V}^+}{\bar{I}^+} = \frac{Z}{\gamma} = \sqrt{\frac{Z}{Y}} \tag{13}$$

Equation 13 describes the characteristic impedance. Using (6) and (7) in Eq. 13 results in

$$\sqrt{\frac{Z}{Y}} = Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad . \tag{14}$$

Hence, the characteristic impedance, $Z_0$, is a complex quantity, i.e., incident voltage, $\bar{V}^+$, and incident current, $\bar{I}^+$, are not in phase in general. However, when no losses are present (R=G=0), then $Z_0$ reduces to a frequency independent resistance, namely $\sqrt{L/C}$ as encountered in transient analysis.

When the negative x directed waves of Eqs. 11 and 12 (reflected waves) are used in Eqs. 4 or 5, then

$$\frac{\bar{V}^-}{\bar{I}^-} = -Z_0 \quad . \tag{15}$$

Hence the equations for total voltage and current are conveniently written as

$$\bar{V}(x) = \bar{V}^+ e^{-\gamma x} + \bar{V}^- e^{+\gamma x} \tag{16}$$

$$I(x) = \frac{\bar{V}^+}{Z_0} e^{-\gamma x} - \frac{\bar{V}^-}{Z_0} e^{+\gamma x} \quad . \tag{17}$$

B. Propagation Constant

Let us now examine the nature of the solutions by looking in detail at the propagation constant $\gamma$. From Eq. 10

$$\begin{aligned}\gamma &= \sqrt{ZY} \\ &= \sqrt{(R+j\omega L)(G+j\omega C)} \\ &= \alpha + j\beta \quad . \end{aligned} \tag{18}$$

Thus, $\gamma$ is a complex number. If now we examine the first term in Eq. 16, one sees

$$\bar{V}^+ \quad e^{-\gamma x} = |\bar{V}^+| \quad e^{j\theta_V^+} e^{-\alpha x} e^{-j\beta x} \quad . \tag{19}$$

The actual voltage as a function of time and space is found by using the procedure given by Eq. 1, e.g.,

$$\begin{aligned} v^+(x,t) &= \sqrt{2}\ \mathrm{Re}\ \bar{V}^+ \quad e^{-\alpha x} e^{j\omega t} e^{-j\beta x} \\ &= \sqrt{2}\ |\bar{V}_x^+|\ e^{-\alpha x} \cos(\omega t - \beta x + \theta_V^+) \quad . \end{aligned} \tag{20}$$

Clearly, Eq. 20 represents a voltage wave traveling in the positive x direction and for our line would be called an incident wave. We note that the amplitude of this wave decays as $e^{-\alpha x}$ as the wave progresses. Hence, $\alpha = \mathrm{Re}\ \gamma$ is called the attenuation constant. This behavior should be expected since now dissipation is present in the line model. The term $\beta = \mathrm{Im}\ \gamma$ is called the phase constant since it represents the rate at which the phase angle in the cosine term changes with distance. That is, for this incident wave, if one holds time fixed and samples the wave phase for increasing x, the phase angle <u>decreases</u> as $-\beta x$. This phase lag is clearly seen by inspection of Fig. 2. Here a cosine wave is traveling toward the right. If time is stopped, and one is at the reference position at the present time, then phase angles measured to the left represent future events at the reference position, while points to the right represent past history and hence correspond to phase lag with respect to the reference position.

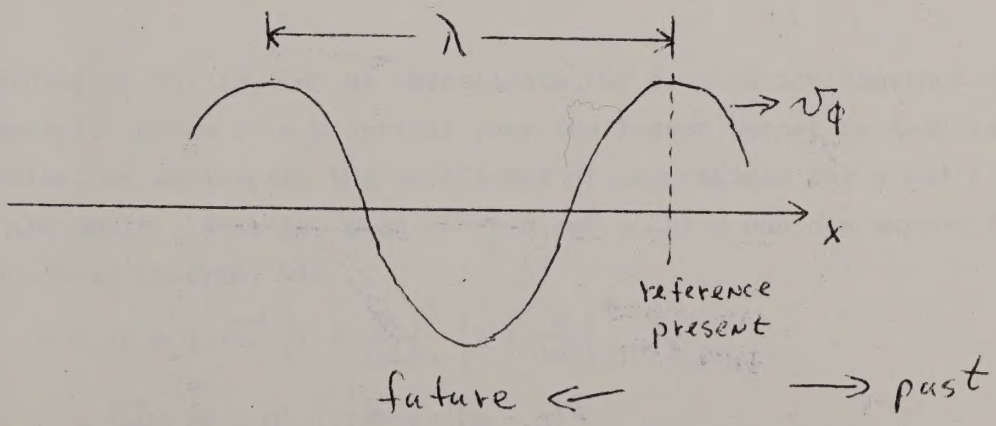



Fig. 2

The phase velocity is easily obtained from Eq. 20 by requiring that the argument of the cosine function remain constant in time. Thus

$$\frac{d}{dt}(\omega t-\beta x+\theta_V^+) = 0$$

or

$$\omega - \beta \frac{dx}{dt} = 0$$

which yields

$$\frac{dx}{dt} = v_\phi = \frac{\omega}{\beta} \text{ m/sec} \quad . \tag{21}$$

That is, x must be changed at the rate of $\omega/\beta$ m/sec to remain at a constant phase point on the traveling wave. This can also be seen by comparing Eq. 20 to our previous general solution in the form

$$v = f(x-vt) \quad . \tag{22}$$

If $\beta$ is factored out of the argument in Eq. 20, then comparison with Eq. 22 clearly indicates that $v = \omega/\beta$.

Figure 2 also indicates that the distance the wave travels in one period, as shown for example as the distance between the crests, is defined as the wavelength. Hence

$$\lambda = v\tau$$
$$= v/f \quad . \tag{23}$$

For free space propagation or air insulated uniform lines, $v = c$, and

$$\lambda = \frac{3\times10^8}{f} \text{ meters} \quad . \tag{24}$$

From Eq. 21, one also sees

$$\omega/\beta = f\lambda \tag{25}$$

or

$$\beta = 2\pi/\lambda \quad . \tag{26}$$

That is, the phase constant always yields a phase change rate of $2\pi$ radians per wavelength. Equations 21, 23 and 26 are encountered frequently when dealing with waves.

Returning to Eq. 18, let us investigate the propagation constant in somewhat more detail. Since in a practical case the losses cannot be made zero at all frequencies, we anticipate the usefulness of expressions for $\alpha$ and $\beta$ when the losses are small. That is, when $\omega L >> R$ and $\omega C >> G$ one can expand (18) using the binomial theorem, viz.,

$$\alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1 + \frac{G}{j\omega C}\right)^{\frac{1}{2}} \quad . \tag{27}$$

Then

$$\alpha \cong \frac{\sqrt{LC}}{2}\left(\frac{R}{L} + \frac{G}{C}\right)\left[1 - \frac{1}{8\omega^2}\left(\frac{R}{L} - \frac{G}{C}\right)^2\right] \tag{28}$$

and

$$\beta \cong \omega\sqrt{LC}\left|1 + \frac{1}{8\omega^2}\left(\frac{R}{L} - \frac{G}{C}\right)^2\right| \quad . \tag{29}$$

Notice that for R = G = 0, lossless lines, or for high frequencies, Eq. 29 yields

$$\beta = \omega\sqrt{LC} \quad . \tag{30}$$

Equation 30 may be obtained directly from (18) with R = G = 0. For this case

$$\frac{\omega}{\beta} = v_\phi = \frac{1}{\sqrt{LC}} \tag{31}$$

as in our previous analysis. Furthermore, R = G = 0 in (28) or directly in (18) yields

$$\alpha = 0 \quad . \tag{32}$$

The lossless line approximation is almost always used for high frequency applications.

C. Distortionless Line

It is very interesting to note from (28) and (29) that for the condition

$$\frac{R}{L} = \frac{G}{C} \tag{33}$$

one obtains

$$\begin{aligned} \beta &= \omega\sqrt{LC} \\ \alpha &\cong \frac{R}{2R_0} + \frac{GR_0}{2} \quad . \end{aligned} \tag{34}$$

Equations 34 indicate that phase velocity and attenuation are independent of frequency. These conditions result in distortionless transmission and are important at lower frequencies, for example on telephone lines. Condition 33 is accomplished in practice by adding inductance at intervals on the line in the form of loading coils. Such lines are called distortionless lines and while the phase constant is the same as a lossless line, the attenuation constant is not zero.

D. Summary

For the line depicted in Fig. 1, the line equations are

$$V = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$$

$$I = \frac{V^+}{Z_0} e^{-\gamma x} - \frac{V^-}{Z_0} e^{+\gamma x}$$

where

$V^+$ = steady state phasor amplitude and phase at x = 0 of the wave traveling in the +x direction

$V^-$ = steady state phasor amplitude and phase at x = 0 of the wave traveling in the -x direction

$\gamma$ = propagation constant given by $\gamma = \alpha + j\beta = \sqrt{ZY}$

$Z$ = series impedance per unit length, $R + j\omega L$

$Y$ = shunt admittance per unit length, $G + j\omega C$

$Z_0$ = characteristic impedance given by $\sqrt{Z/Y}$.

For lossless lines or high frequency excitation

$$\gamma = j\beta = j\omega\sqrt{LC}$$

$$\alpha = 0$$

$$Z_0 = R_0 = \sqrt{L/C} \quad .$$

In addition, it is always true that

$$v = \omega/\beta$$

$$f = v_\phi/\lambda$$

and

$$\beta = 2\pi/\lambda \quad .$$

E. Transformation of Reference

As will become apparent, the measurement of transmission line length using the load as a reference is far more convenient for steady state excitation than placing the reference at the generator. To this end we rewrite the line equations

$$V = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$$
$$I = \frac{V^+}{Z_0} e^{-\gamma x} - \frac{V^-}{Z_0} e^{+\gamma x} , \tag{35}$$

and using Fig. 3, wherein the distance variable S has its zero reference at the load, make the substitution $x = \ell - S$ in Eqs. 35. Hence, for example,

$$V = V^+ e^{-\gamma \ell} e^{\gamma S} + V^- e^{+\gamma \ell} e^{-\gamma S}$$
$$= V_1 e^{+\gamma S} + V_2 e^{-\gamma S} \tag{36}$$

where

$$V_1 = V^+ e^{-\gamma \ell} , \qquad V_2 = V^- e^{+\gamma \ell} . \tag{37}$$

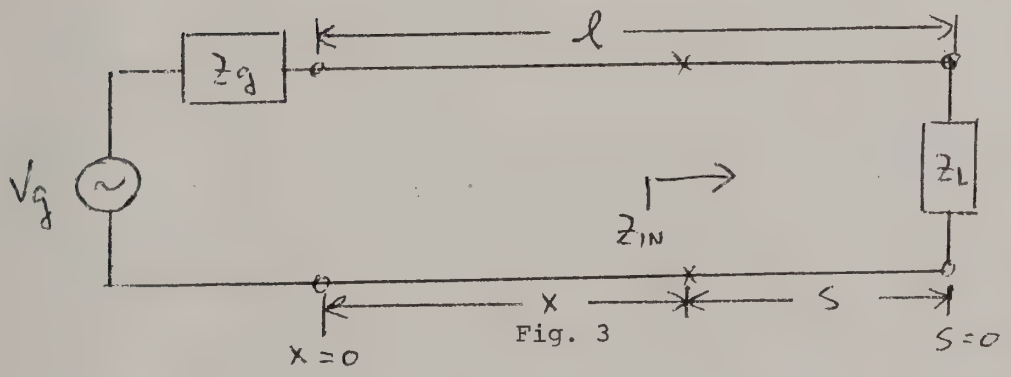


Fig. 3

Thus, the first term in (36) represents an incident wave on the line since it travels in the negative S direction. We use phasor coefficients $V_1$ and $V_2$ in the S system, and they are related to $V^+$ and $V^-$ through Eq. 37.

Let us now describe the input impedance looking to the right at point x in Fig. 3. The ratio of total voltage to total current at this point can be written as

$$Z_{IN}(S) = \frac{V(S)}{I(S)} = Z_0 \left[ \frac{V^+ e^{-\gamma \ell} e^{\gamma S} + V^- e^{\gamma \ell} e^{-\gamma S}}{V^+ e^{-\gamma \ell} e^{\gamma S} - V^- e^{\gamma \ell} e^{-\gamma S}} \right]$$
$$= Z_0 \left[ \frac{1 + \frac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}} e^{-2\gamma S}}{1 - \frac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}} e^{-2\gamma S}} \right] . \tag{38}$$

Let us examine the cumbersome term $V^- e^{\gamma \ell}/V^+ e^{-\gamma \ell}$ that, according to Eq. 37, is just $V_2/V_1$ and would appear to be a reflection coefficient since it is the ratio of reflected to incident voltage at the load (i.e., at $x = \ell$ or $S = 0$). This can easily be seen by expressing the load voltage and current by both line equations and load equations (Eqs. 35 with $x = \ell$)

$$\begin{aligned} V_L &= V^+ e^{-\gamma \ell} + V^- e^{+\gamma \ell} \\ I_L &= \frac{V^+}{Z_0} e^{-\gamma \ell} - \frac{V^-}{Z_0} e^{+\gamma \ell} \\ \frac{V_L}{I_L} &= Z_L \quad . \end{aligned} \tag{39}$$

Combining Eqs. 39 yields

$$\begin{aligned} \frac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}} &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= K_L \quad . \end{aligned} \tag{40}$$

One sees again that the reflection coefficient, which is the ratio of reflected to incident voltage is determined entirely by load conditions and is independent of line length. Notice also that the reflection coefficient is a complex number.

We may now reexpress Eq. 38 as

$$Z_{IN}(S) = Z_0 \left[ \frac{1 + K_L e^{-2\gamma S}}{1 - K_L e^{-2\gamma S}} \right] \tag{41}$$

wherein one sees that only load conditions, which determine $K_L$, and distance from the load to the point at which $Z_{IN}$ is desired are of importance. This is why a distance reference at the load is so useful, and why we will use the S system for all steady state measurements. Hence the line equations will be written as (36) and its companion current equation

$$V(S) = V_1 e^{\gamma S} + V_2 e^{-\gamma S} \tag{42}$$

$$I(S) = \frac{V_1}{Z_0} e^{\gamma S} - \frac{V_2}{Z_0} e^{-\gamma S} \tag{43}$$

and

$$K_L = \frac{V_2}{V_1} \quad . \tag{44}$$

We note from Eq. 42 that the terms represent incident voltage at S and reflected voltage at S respectively. Hence the ratio of these terms must be by definition the reflection coefficient at point S, i.e.,

$$\frac{V_2 e^{-\gamma S}}{V_1 e^{+\gamma S}} = K(S) \quad . \tag{45}$$

Then, using (44)

$$K(S) = K_L e^{-2\gamma S} \quad . \tag{46}$$

Hence, the reflection coefficient at any point S is related to the reflection coefficient at the load by the factor $e^{-2\gamma S}$. Inspection of Eq. 41 indicates that an easily remembered equation for input impedance is

$$Z_{IN}(S) = Z_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \quad . \tag{47}$$

F. Determination of the Line Equation Coefficients

The complex coefficients $V_1$ and $V_2$ in Eqs. 42 and 43 represent the unknown coefficients in the solution of a differential equation. One would thus expect that they can be determined by examination of the boundary conditions imposed on the equation solution. At the load boundary we have already determined through Eqs. 40 and 44 that

$$\frac{V_2}{V_1} = \frac{Z_L - Z_0}{Z_L + Z_0} \quad ; \tag{48}$$

at the generator boundary

$$V_{IN} = \frac{V_g Z_{IN}}{Z_{IN} + Z_g} \quad . \tag{49}$$

Using $s = \ell$ in Eq. 42 then yields

$$V_1 e^{\gamma\ell} + V_2 e^{-\gamma\ell} = \frac{V_g Z_{IN}}{Z_{IN} + Z_g} \quad . \tag{50}$$

Equations 48 and 50 serve as two equations in two unknowns from which the complex coefficients $V_1$ and $V_2$ can be determined. In most practical problems the equations can be applied in the form given and serve as the starting point for transmission line analysis. It is to be emphasized at this point that $V_1$, for example, <u>is not</u> determined by input conditions alone but rather is a steady state coefficient which depends on <u>both</u> input and load conditions. To clarify this point, we will digress briefly to show the relationship between steady state and transient analysis.

G. Relationship Between Steady State and Transient Analysis

Using Eq. 41 in (50) yields

$$V_1 e^{\gamma\ell} + V_2 e^{-\gamma\ell} = \frac{V_g Z_0 (1+K_L e^{-2\gamma\ell})}{Z_0(1+K_L e^{-2\gamma\ell}) + Z_g(1-K_L e^{-2\gamma\ell})} \quad . \tag{51}$$

Factoring $V_1 e^{\gamma\ell}$ out of the left side and using (44) results in

$$V_1 e^{\gamma\ell}(1+K_L e^{-2\gamma\ell}) = \frac{V_g Z_0 (1+K_L e^{-2\gamma\ell})}{Z_g + Z_0 - (Z_g - Z_0) K_L e^{-2\gamma\ell}} \quad . \tag{52}$$

Dividing numerator and denominator on the right by $Z_g + Z_0$ then gives

$$V_1 = \frac{\frac{V_g Z_0}{Z_g+Z_0} e^{-\gamma \ell}}{1 - K_L\left(\frac{Z_g-Z_0}{Z_g+Z_0}\right) e^{-2\gamma \ell}} \quad . \tag{53}$$

Equation 53 gives the complex incident phasor coefficient in terms of generator, load, and line conditions.

Let us now attempt to determine $V_1$ from the point of view of transient analysis used earlier. Imagine a switch in series with the generator of Fig. 3 which is closed at t = 0. At the instant excitation is applied, the input voltage is given by

$$V_{IN(0^+)} = V^+(0^+) = \frac{V_g Z_0}{Z_g + Z_0} \quad . \tag{54}$$

However, the voltage will not remain at this value since a reflection returning from the load will soon modify it. If neither the load nor the generator are matched, this reflection process will continue and a general expression for the incident voltage will be

$$V^+ = \frac{V_g Z_0}{Z_g + Z_0} + \frac{V_g Z_0}{Z_g + Z_0} K_L K_g e^{-2\gamma \ell} + \frac{V_g Z_0}{Z_g + Z_0} (K_L K_g e^{-2\gamma \ell})^2 + \ldots$$

$$= \frac{V_g Z_0}{Z_g + Z_0} \sum_{n=0}^{\infty} [K_L K_g e^{-2\gamma \ell}]^n \quad . \tag{55}$$

The infinite series in (55) can be summed with the result

$$V^+ = \frac{V_g Z_0}{Z_g + Z_0} \frac{1}{1 - K_L K_g e^{-2\gamma \ell}} \quad . \tag{56}$$

Using Eq. 37 for $V^+$ we see that

$$V_1 = \frac{\frac{V_g Z_0}{Z_g+Z_0} e^{-\gamma \ell}}{1 - K_L K_g e^{-2\gamma \ell}} \quad . \tag{57}$$

Considering that

$$K_g = \frac{Z_g - Z_0}{Z_g + Z_0} \quad , \tag{58}$$

we see that Eqs. 57 and 53 are identical. We have thus determined the coefficient $V_1$ from two entirely different points of view.

Thus, when Eqs. 48 and 50 are used to determine $V_1$ and $V_2$, generator conditions are automatically included and there is no need to consider a "reflection coefficient at the generator." This latter concept, in fact, has no meaning in the steady state since the "steady state" means all transient effects have died out.

## II. Application to Lossless Lines

### A. Analytical Method

In order to simplify the equations developed in Section I and clearly illustrate the theory by application, lossless line examples will be considered. As mentioned previously even though lossless lines are a special case they represent a major area of application since nearly all high frequency lines may justifiably be considered lossless. For lossless lines (R=G=0)

$$Z_0 = R_0 = \sqrt{L/C}$$
$$\gamma = j\beta = j\omega\sqrt{LC}$$

and the line equations become

$$V = V_1 e^{j\beta S} + V_2 e^{-j\beta S} \tag{59}$$

$$I = \frac{V_1}{R_0} e^{j\beta S} - \frac{V_2}{R_0} e^{-j\beta S} \quad . \tag{60}$$

Equations 59 and 60 yield in turn

$$Z_{IN}(S) = \frac{V(S)}{I(S)} = R_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \tag{61}$$

where

$$K(S) = K_L e^{-2j\beta S} \tag{62}$$

and

$$K_L = |K_L| e^{j\phi_L} = \frac{Z_L - R_0}{Z_L + R_0} \quad . \tag{63}$$

Notice that on a lossless line the magnitude of K(S) is always constant and equal to $|K_L|$, only the phase angle changes with distance.

Consider the following problem. A steady state sinusoidal source with internal resistance of 100 Ω is connected to a 3λ/4 50 Ω line driving a real load of 25 Ω as depicted in Fig. 4. We seek the voltage and current at all points S along the line. To find the phasor coefficients in Eqs. 59 and 60 we apply the boundary conditions. From (48)

$$\frac{V_2}{V_1} = K_L = \frac{Z_L - R_0}{Z_L + R_0} = \frac{25 - 50}{25 + 50} = 0.33\, e^{j\pi} \quad . \tag{64}$$

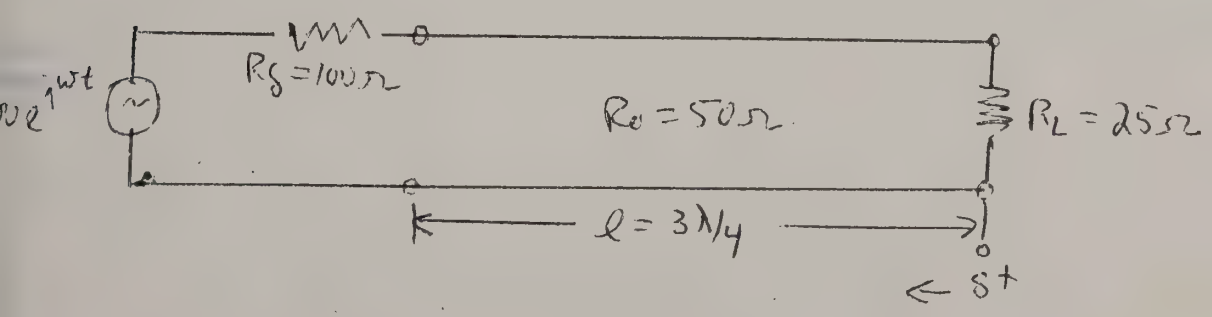



Fig. 4

At the input Eq. 61 gives

$$Z_{IN}(\ell) = R_0\left[\frac{1 + K(\ell)}{1 - K(\ell)}\right] \quad . \tag{65}$$

With $K(\ell)$ given by (62),

$$\begin{aligned} K(\ell) &= K_L\, e^{-j2\beta\ell} \\ &= (0.33\, e^{j\pi})(e^{-j2(2\pi/\lambda)(3\lambda/4)}) \\ &= 0.33 \ , \end{aligned} \tag{66}$$

Eq. 65 may be evaluated to yield

$$Z_{IN} = 50\left[\frac{1 + 0.33}{1 - 0.33}\right] = 100\ \Omega \quad . \tag{67}$$

Hence, the input boundary condition, Eq. 50, is

$$V_1\, e^{j\beta\ell} + V_2\, e^{-j\beta\ell} = \frac{V_g\, Z_{IN}}{Z_{IN} + Z_g}$$

or

$$V_1\, e^{j(3\pi/2)} + V_2\, e^{-j(3\pi/2)} = \frac{200(100)}{100 + 100}$$

$$j(V_2 - V_1) = 100 \quad . \tag{68}$$

Solving (68) simultaneously with (64) finally yields the desired coefficients,

$$V_1 = j\ 75 = 75\, e^{j\pi/2} \ \text{volts} \tag{69}$$

$$V_2 = -j\ 25 = 25\, e^{-j\pi/2} \ \text{volts} \quad . \tag{70}$$

Let us now examine the voltage solution in some detail. Equation 59 becomes

$$V(S) = 75\, e^{j\pi/2}\, e^{j\beta S} + 25\, e^{-j\pi/2}\, e^{-j\beta S} \quad . \tag{71}$$

Evaluating (71) at the load yields

$$V_L = 75\, e^{j\pi/2} + 25\, e^{-j\pi/2} = 50\, e^{j\pi/2} \tag{72}$$

while at the input (71) gives

$$V_{IN} = 75\, e^{j2\pi} + 25\, e^{-j2\pi} = 100 \quad . \tag{73}$$

Equations 72 and 73 are plotted in Fig. 5 to clearly illustrate that the individual components of the voltage, i.e., incident or reflected, when compared at input and load ends always have a phase difference equal to the line phase shift, the total voltage in general does not. Only when the line is matched or in certain very special cases will the total voltage at any two points have a phase difference equal to that of the line phase shift.

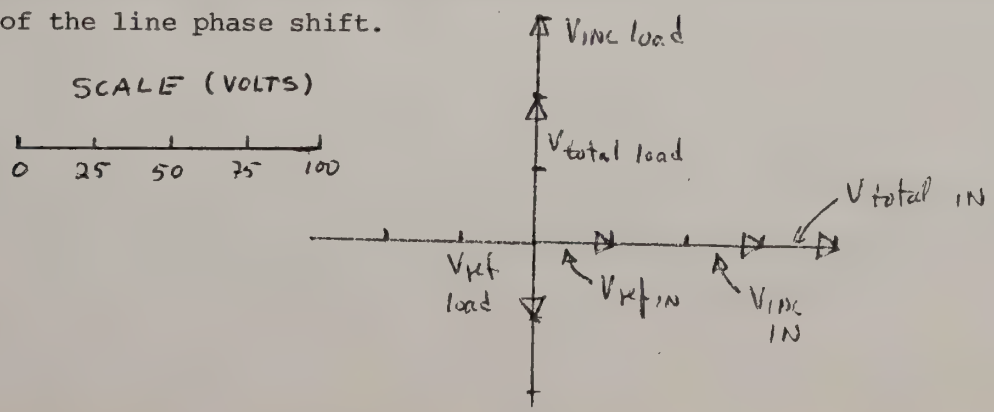



Fig. 5

B. Graphical Method

It is not always necessary to determine the coefficients $V_1$ and $V_2$. In many applications a knowledge of the relative voltage and current variation is sufficient. In addition, in practice a voltage or current at some point may be known from measurment. Hence with such information added to the relative variation, the voltage and current become known at all points quantitatively.

Consider again Eq. 59 factored to read

$$V = V_1 e^{j\beta S}\left[1 + \frac{V_2}{V_1} e^{-2j\beta S}\right] \quad . \tag{74}$$

Using (62) and (44) allows the voltage to be written as

$$\begin{aligned} V &= V_1 e^{j\beta S}[1 + K(S)] \\ &= V_1 e^{j\beta S}[1 + |K_L| e^{j\phi L} e^{-2j\beta S}] \quad . \end{aligned} \tag{75}$$

Since the relative magnitude of the total voltage is determined entirely by the bracketed term it is useful to consider it alone. It is clear from (63) that for all possible $Z_L$

$$0 \leq |K_L| \leq 1 \quad . \tag{76}$$

Thus the term may be conveniently examined graphically by the plot known as the "crank diagram" shown in Fig. 6. It is evident from (76) that all possible points must lie within a circle of radius 1. Hence one plots the complex number $K_L$ in polar coordinates and then decreases its angle by $2\beta S$ when moving back along the line toward the generator as demanded by Eq. 75. If we now

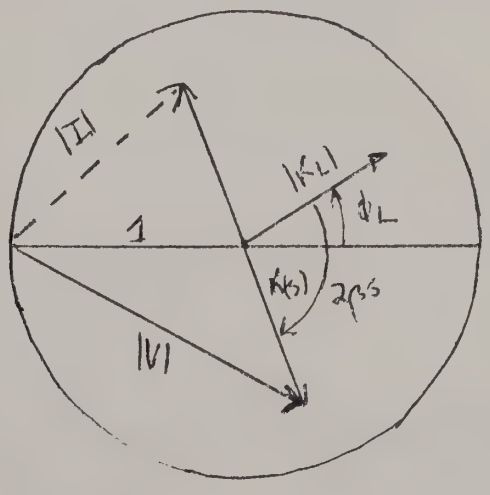



Fig. 6. A plot of $1 + \bar{K}(S)$.

draw a vector diagram using the left end of the diameter shown in Fig. 6 as a starting point, then the left half of the diameter is the vector $1/0°$ and when K(S) is vectorially added to it we have the sum indicated in the bracketed term

of (75). Hence, the voltage magnitude variation is given by the length of the vector joining the starting point reference to the tip of the general reflection coefficient K(S). Motion on the line cooresponds to rotation of K(S) hence the name crank diagram. One should note that relative phase angle can also be obtained from this diagram but the term $e^{j\beta S}$ in Eq. 75 must also be included. It should also be apparent from comparison of Eqs. 59 and 60 that a vector drawn from the starting point reference to the tip of the vector -K(S) will yield the relative current magnitude along the line.

Suppose now we make such a plot for the example already considered. From Eq. 64 we have the necessary information to determine the plot. The reflection coefficient of the load, $K_L$, from (64) is shown plotted in Fig. 7. The variation of K(S) is now seen to include, starting at the reference point S = 0, one and one-half revolutions on the polar chart. The total voltage magnitude, Eq. 75, is easily obtained as the length of the line joining the reference point

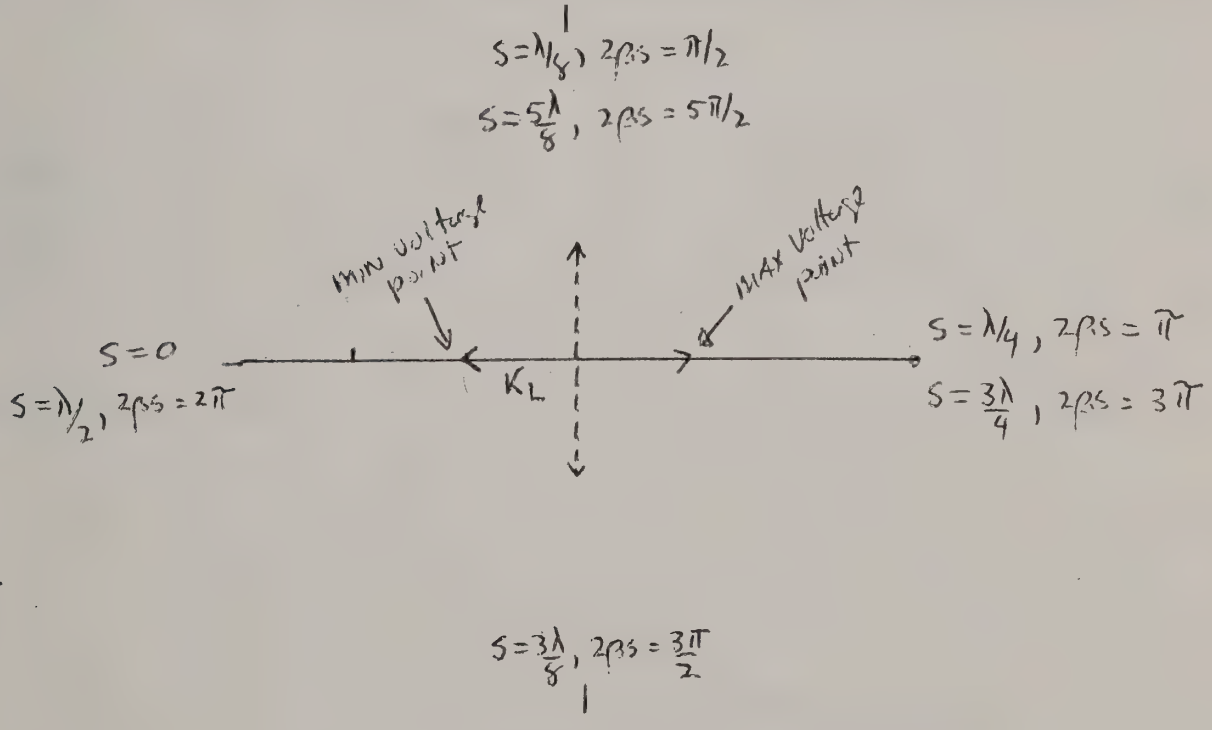



Fig. 7

with the tip of the rotating reflection coefficient vector. A plot of the relative voltage is shown in Fig. 8, notice that for this problem the actual voltage magnitude at all points is readily obtained since the voltage at the input, namely 100 volts, is known.

The current standing wave is also shown in Fig. 8 and was obtained from Fig. 7 through the construction

$$|I| \propto |1 - K(S)| \quad . \tag{77}$$

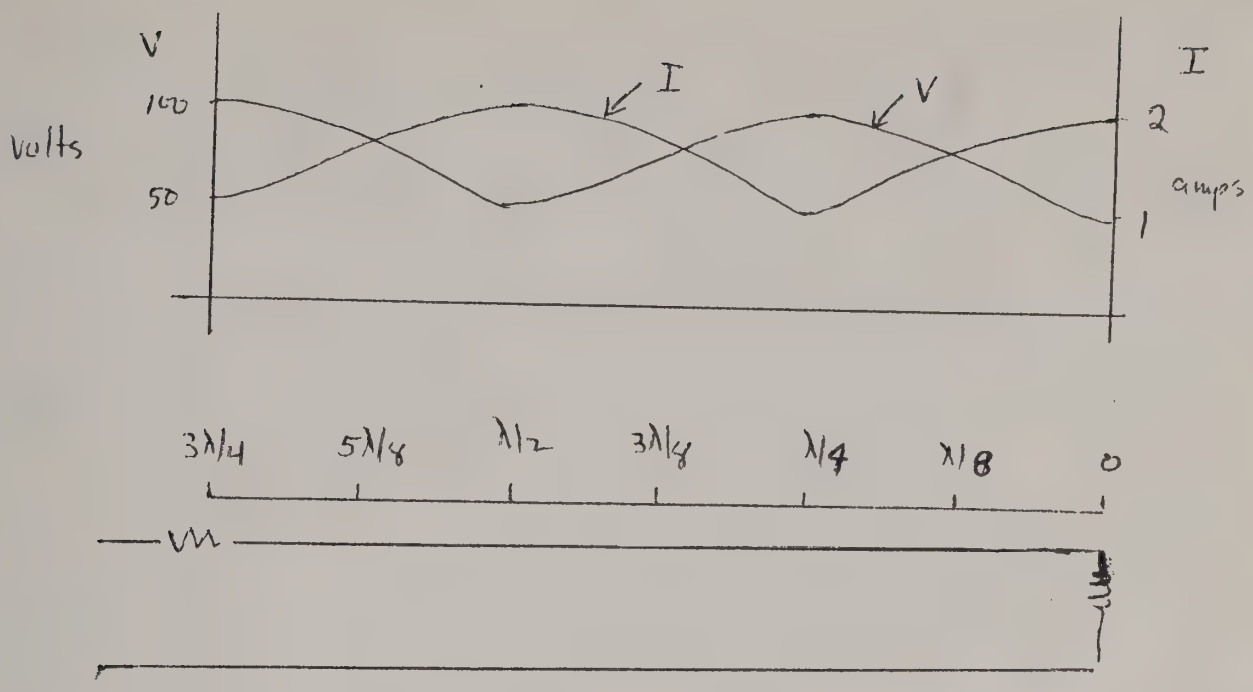



Fig. 8

This construction, of course, gives the relative current variation, however knowledge of the voltage and the load impedance allows calculation of the total current magnitude. One notes from Fig. 8 that a current minimum occurs at the position of a voltage maximum and vice-versa. That this will <u>always</u> be true should be obvious from the constructions in Fig. 7.

The curves in Fig. 8 are called standing waves. The name implies that they are stationary in space as opposed to traveling waves. Indeed, they are each the <u>result</u> of <u>two</u> traveling waves in opposite directions. Recall that the curves represent phasor magnitude hence the time varying voltage and current associated with them does have amplitude variation, however all points change by the same factor so that the relative positions of maximum, minimum, etc., do not move in space.

The ratio of maximum voltage to minimum voltage is defined as the voltage standing wave ratio, and from the diagram of Fig. 7 is given by

$$\text{VSWR} = \frac{V_{max}}{V_{min}} = \frac{1 + |K_L|}{1 - |K_L|} \quad . \tag{78}$$

For this exaple VSWR = 2 which is also indicated by Fig. 8. Since we have seen earlier that

$$K_L = V_2/V_1$$

it is then true that

$$|K_L| = |V_2|/|V_1| \quad . \quad (79)$$

Using (79) in (78) yields

$$VSWR = \frac{|V_1| + |V_2|}{|V_1| - |V_2|} \quad . \quad (80)$$

Comparing Eq. 80 with Eq. 59 for the total voltage leads one to the conclusion that at a voltage maximum the incident and reflected voltage waves must be adding in phase, while at a minimum these waves are 180° out of phase. Furthermore, Eq. 60 indicates that at a voltage maximum the current is given by

$$I = \frac{|V_1|}{R_0} - \frac{|V_2|}{R_0} \quad , \quad (81)$$

hence, the voltage and current are in phase and the input impedance must be real. The same is true for a voltage minimum. We shall return to these conclusions shortly.

The standing waves of Fig. 8 are not sinusoids, and, in general, standing waves are not sinusoids. This can most easily be seen by examining the magnitude of the bracketed term in Eq. 75, i.e., $|V|$ normalized with respect to $|V_1|$.

$$\frac{|V|}{|V_1|} = \left|1 + |K_L|\, e^{j\theta}\right| \quad (82)$$

where

$$\theta = \phi_L - 2\beta S \quad .$$

The desired magnitude in (82) becomes

$$\frac{|V|}{|V_1|} = \sqrt{(1+|K_L|\cos\theta)^2 + (|K_L|\sin\theta)^2}$$

$$= \sqrt{1 + |K_L|^2 + 2|K_L|\cos\theta} \quad . \quad (83)$$

Clearly Eq. 83 does not represent a sinusoidal variation of magnitude with S. This should also be apparent from the crank diagram of Fig. 7 wherein one sees that in general, the minima are sharper than the maxima.

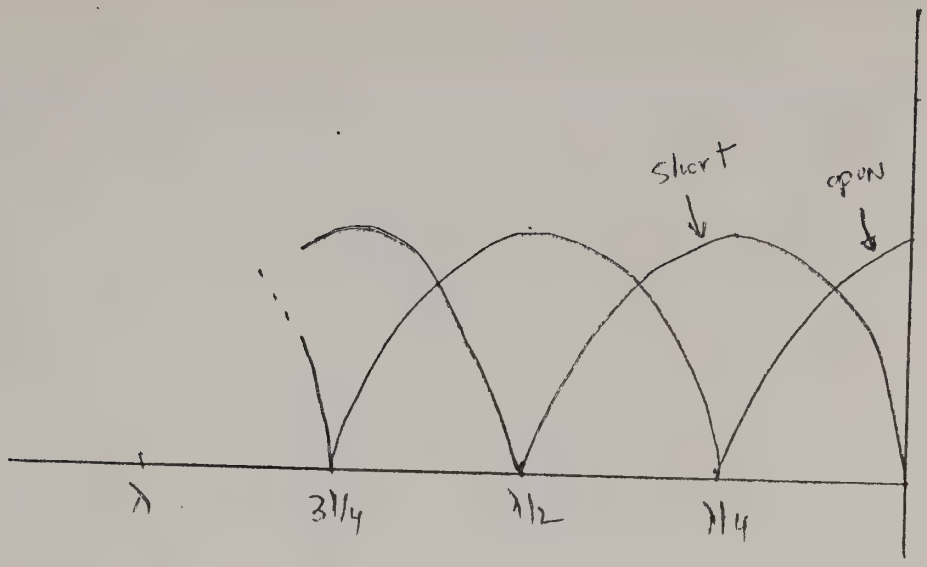
Equation 83 does represent a sinusoid for two special cases, an open circuited and a short circuited line. For the open circuit, $|K_L| = 1$ and $\theta = -2\beta S$, hence, Eq. 83 yields

$$\frac{|V|}{|V_1|} = \sqrt{2 + 2\cos 2\beta S} = 2\,|\cos\beta S| \quad . \quad (84)$$

For the short circuit, $|K_L| = 1$ and $\theta = \pi - 2\beta S$, hence, Eq. 83 now gives

$$\frac{|V|}{|V_1|} = 2\,|\sin\beta S| \quad . \quad (85)$$

These results are plotted in Fig. 9 where one sees that VSWR = ∞ for both cases.


short
open
λ
3λ/4
λ/2
λ/4

C. Instantaneous Standing Wave

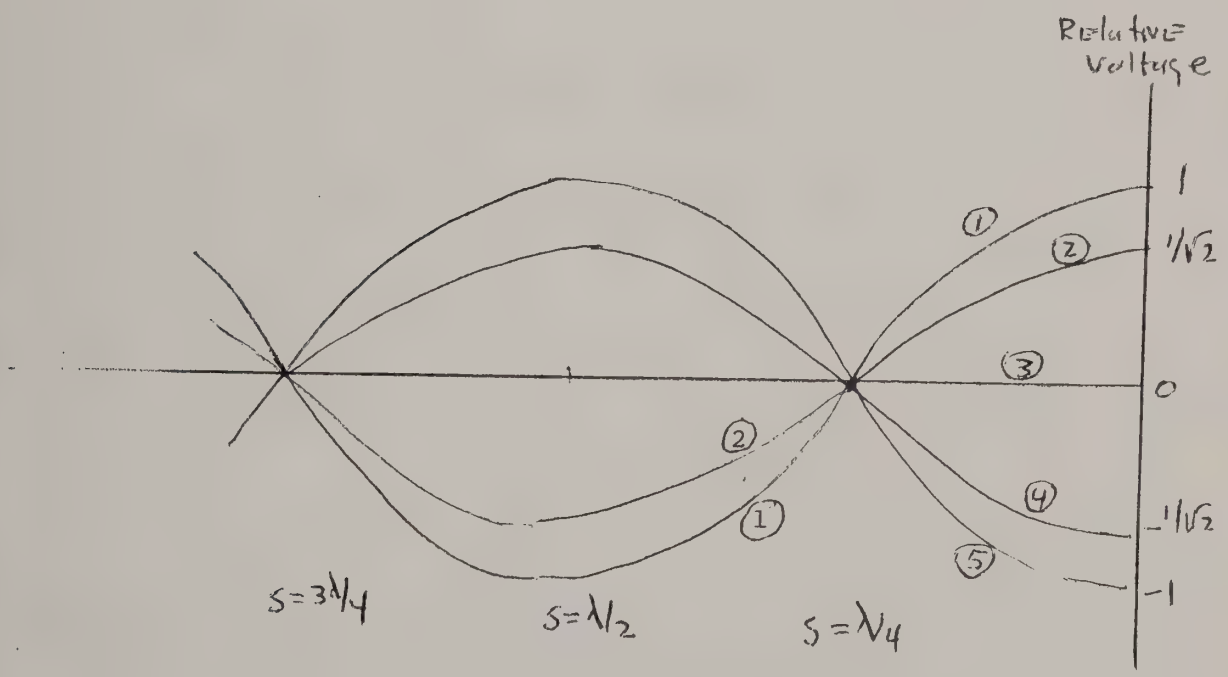
The phasor expressions for the standing waves developed to this point only give the envelope of the instantaneous standing wave, i.e., the reading of an ac meter. Let us take, for example, the open circuited line case and examine the line voltage as a function of time and space. Given that

$$V(S) = 2V_1 \cos\beta S \text{ rms volts} \quad , \tag{86}$$

we determine

$$v(S,t) = 2\sqrt{2}\,|V_1| \cos\beta S \cos(\omega t+\theta_1) \quad . \tag{87}$$

Thus, it is seen that the instantaneous voltage has the same spacial dependence, however, its amplitude varys with time. Equation 87 is shown plotted in Fig. 10 for five values of $(\omega t+\theta_1)$. Notice the angle $\theta_1$ from the line voltage coefficient $V_1$ merely shifts the time reference. Clearly the maxima and minima are stationary in space and the rms envelope of the instantaneous voltage is the phasor standing wave.




1 $(\omega t+\theta_1) = 0$

2 $(\omega t+\theta_1) = \pi/4$

3 $(\omega t+\theta_1) = \pi/2$

4 $(\omega t+\theta_1) = 3\pi/4$

5 $(\omega t+\theta_1) = \pi$

Fig. 10

D. Input Impedance, Graphical Considerations

It has been shown that the input impedance can be described by

$$Z_{IN}(S) = R_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \quad . \tag{88}$$

One notes that Eq. 88 is just the ratio of V(S) to I(S). Since the numerator and denominator have each been interpreted graphically it is a straightforward matter to extend these ideas to impedance. Equation 88 demands a one to one correspondence between K(S) and $Z_{IN}(S)$ so the plot of K(S) used previously can be used to determine impedance. For example, from Fig. 11 which shows K(S) and -K(S) one can find the magnitude and angle of the complex $Z_{IN}(S)$ by forming the ratio of the vectors 1 + K(S) and 1 - K(S). While this is useful we shall see shortly that a related plot, the Smith chart, is a more convenient approach for obtaining quantitative results.

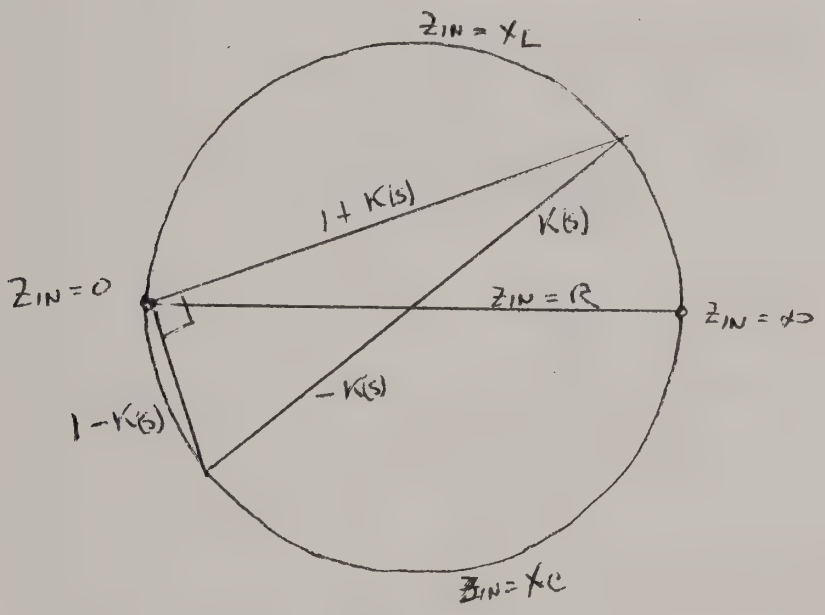



Fig. 11

However, from Fig. 11 it is quite helpful for understanding the Smith chart to recognize that, as dictated by Eq. 88, the horizontal diameter of the circle must correspond to real values of $Z_{IN}$ since the angles of 1 + K(S) and 1 - K(S) are both zero. (It is also interesting to note that all points on the horizontal diameter correspond to either minimum or maximum voltage of the standing wave.) In addition, the left hand end of the diameter must be $Z_{IN} = 0$ while the right hand end is $Z_{IN} = \infty$. It can further be said that all $Z_{IN}$ lying within the upper semicircle will have positive angle while those in the lower semicircle will have negative angle, and, in fact all points on the upper circumference will have angle +90° ($X_L$), while those on the lower circumference will have angle -90° ($X_C$). Since a one to one correspondence between K(S) and $Z_{IN}(S)$

exists (Eq. 88), and since we know that a full rotation of K(S) on the crank diagram corresponds to a distance $S_r$ such that

$$2\beta S_r = 2\pi \text{ radians}$$

or

$$S_r = \frac{\pi}{\beta} = \frac{\lambda}{2} \text{ meters} ,$$

we must conclude that $S_r$ represents the impedance repetition length. That is, the impedance must repeat at $\lambda/2$ intervals since this distance corresponds to a complete rotation on the crank diagram. One concludes that a knowledge of the impedance within a half wavelength of line is tantamount to a knowledge of impedance everywhere on the line regardless of its length. These considerations will be expanded upon when the Smith chart is discussed.

E. Determination of Unknown Impedance

At high frequencies it is often impossible to measure impedance with a bridge or by other low frequency techniques. Such high frequency measurements are conveniently made using transmission lines. For example, suppose it is desired to determine an unknown load impedance connected to a transmission line. Equation 88 specialized to the load position yields

$$Z_L = R_0\left(\frac{1 + K_L}{1 - K_L}\right) . \tag{89}$$

Hence to determine $Z_L$ it is necessary to determine the magnitude and angle of $K_L$.

We have already seen in Eq. 78 that the magnitude of $K_L$ is related to the standing wave ratio, so solving (78) for $|K_L|$

$$|K_L| = \frac{VSWR - 1}{VSWR + 1} . \tag{90}$$

Hence, by a relatively straightforward measurement of VSWR on the line half our problem is solved. One should note that quantities such as VSWR lend themselves to easy measurement since no absolute voltage calibration is necessary to measure ratio. Consideration of the crank diagram will show how the angle $\phi_L$ can be measured. Figure 12 shows a line and its associated crank diagram. The distance $S_{MIN}$ is the distance from the load to the first voltage minimum on the line, and its corresponding relation to the crank diagram is also shown. From the diagram it can be seen that

$$\phi_L - 2\beta S_{MIN} = -\pi \tag{91}$$

so that

$$\phi_L = 2\beta S_{MIN} - \pi . \tag{92}$$

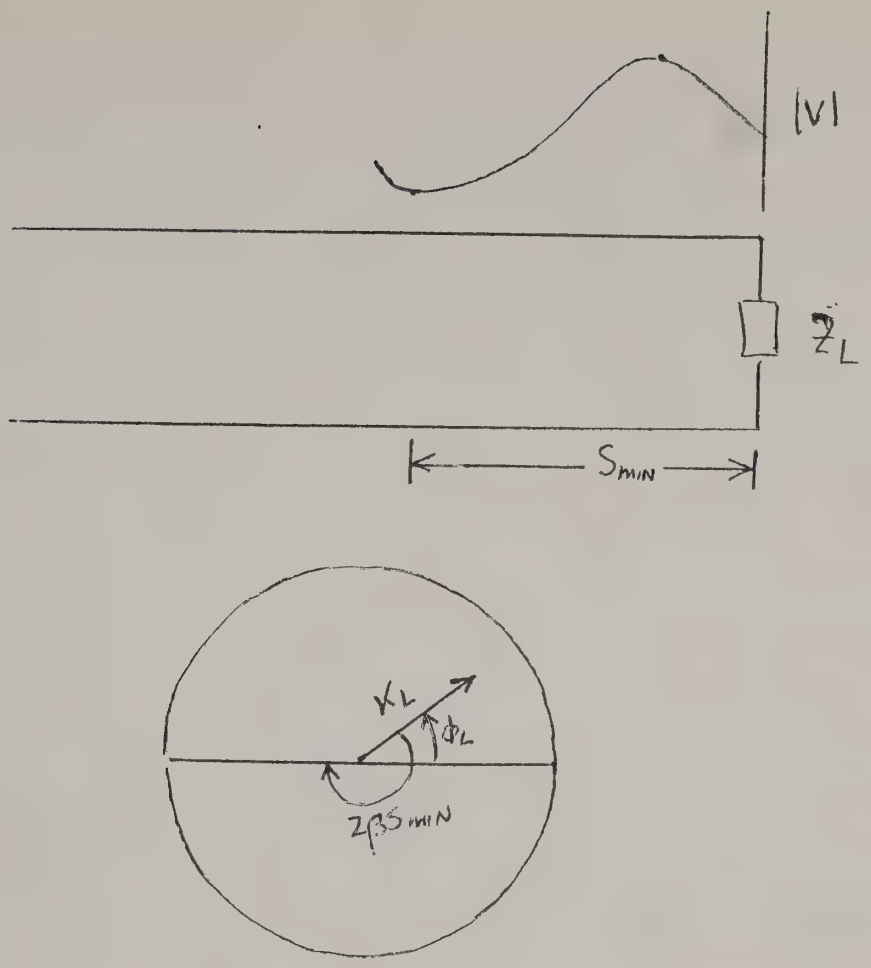



Fig. 12

Equations 91 and 92 thus allow determination of the complex reflection coefficient of the load by two easily performed measurements namely, VSWR and distance to the first minimum voltage from the load.

With a knowledge of complex $K_L$ obtained as outlined, it is now possible to determine the unknown $Z_L$ through use of Eq. 89. It must be emphasized at this point that not only has the load impedance been determined, but because $K(S) = K_L e^{-2j\beta S}$ in Eq. 88 is also now available for any S, the impedance at every position on the line is now known.

F. Power Considerations

As has been pointed out earlier, the "incident wave" of steady state analysis is not the same as the incident wave of transient analysis. Although the two can be related since the steady state wave is the sum of all the transient incident waves, it is incorrect to imagine that the steady state and transient incident waves are one and the same. Now since a transient analysis incident wave is in fact a localization of energy or at least an "energy front," it is valid to associate a power flow with such an incident wave. Hence, the

trap is set to do the same with a steady state incident wave and herein lies the problem. It will now be shown that in the steady state, the association of power flow as so-called "incident power" or "reflected power" has limited physical meaning. While these terms may have usefulness, they are mathematical concepts and great care must be exercised in their application.

Consideration of the expressions for voltage and current on a transmission line naturally leads one to imagine that the product of incident voltage and incident current must be the incident power with the same ideas holding true for the reflected waves, i.e.,

$$P_{INC} = V_{INC}\, I^*_{INC} = \frac{|V_1|^2}{R_0} \tag{93}$$

$$P_{ref} = V_{ref}\, I^*_{ref} = -\frac{|V_2|^2}{R_0} = -\frac{|K_L|^2\,|V_1|^2}{R_0} \quad . \tag{94}$$

Obviously the complex powers in Eqs. 93 and 94 turn out to be real so some special interpretation must be in order. (That they are real is no surprise, since on a lossless line $V_{INC}$ is always in phase with $I_{INC}$ because the characteristic impedance is real.)

Let us examine the following problem to see the danger in assigning a physical meaning to these individual powers. Consider a transmitter driving a transmission line connected to an antenna. Suppose some mismatch occurs at the antenna, a common situation, then according to (94) a reflected power is created. What now becomes of this power? Is it absorbed in the output impedance of the transmitter? Suppose the output impedance is zero, then what? It is not the "reflected power" that is causing the problem, but rather the concept that reflected power represents average power flow towards the input.

We now calculate properly the transmission line power by forming the product of <u>total</u> voltage with the conjugate of <u>total</u> current, viz.

$$\begin{aligned} P &= VI^* \\ &= (V_1 e^{j\beta S} + K_L V_1 e^{-j\beta S}) \left(\frac{V_1 e^{j\beta S}}{R_0} - \frac{V_1 K_L e^{-j\beta S}}{R_0}\right)^* \\ &= (V_1 e^{j\beta S} + K_L V_1 e^{-j\beta S}) \left(\frac{V_1^* e^{-j\beta S}}{R_0} - \frac{V_1^* K_L^* e^{+j\beta S}}{R_0}\right) \\ &= \frac{|V_1|^2}{R_0} - \frac{|K_L|^2\,|V_1|^2}{R_0} + \frac{|K_L|\,|V_1|^2}{R_0} e^{-j(2\beta S-\phi_L)} - \frac{|K_L|\,|V_1|^2}{R_0} e^{+j(2\beta S-\phi_L)} \\ &= \frac{|V_1|^2}{R_0}(1 - |K_L|^2) - j\,\frac{2|V_1|^2\,|K_L|}{R_0}\sin(2\beta S-\phi_L) \quad . \end{aligned} \tag{95}$$

Clearly the average power is

$$P_{AVG} = \text{Re } VI^* = \frac{|V_1|^2}{R_0}(1 - |K_L|^2) \quad . \tag{96}$$

By comparing with Eqs. 93 and 94 one sees that indeed the average power is the difference between the "incident" and "refelcted" powers. The average power has physical meaning, but to focus one's attention on either term in the average power to the exclusion of the other may be erroneous and lead to unanswerable questions such as posed earlier.

The reactive power is obtained from (95) by

$$P_{react} = \text{Im } VI^* = -\frac{2|V_1|^2\,|K_L|}{R_0}\sin(2\beta S - \phi_L) = \frac{2|V_1|^2\,|K_L|}{R_0}\sin(\phi_L - 2\beta S) \quad . \tag{97}$$

This term represents the reactive power delivered to the load and to energy storage on the line. Notice that the angular dependence in the argument of the sine function is identical to that in the standing waves. For a real load which accepts no reactive power but is mismatched to the line one notes that the reactive energy storage still exists and a net reactive power flow is found at each point. When the line is matched, $|K_L| = 0$, and the reactive power flow term disappears. Energy is still stored on the line, but now inductive and capacitive storage are equal and the net reactive power flow is zero.

One final point may be made from the observation of Eq. 96. It appears that $|K_L|^2$ could be considered a power reflection coefficient. Indeed this is useful for making average power calculations but the term $1 - |K_L|^2$ is the one with physical meaning and not $|K_L|^2$ alone.

ISL ORIGINALS



## The Smith Chart

### A. Theory

The expression for the input impedance of a time harmonically excited lossless transmission line has been derived as

$$Z_{IN}(S) = R_0 \left| \frac{1 + K(S)}{1 - K(S)} \right| \tag{1}$$

where

$$K(S) = |K_L|\, e^{j\phi L}\, e^{-j2\beta S} \quad .$$

Thus, as has been pointed out, a one to one correspondence exists between reflection coefficient at some point S, K(S), and the input impedance at that point. Since we have seen from the crank diagram that all possible reflection coefficients for any given line can be plotted within a circle of unity radius it must be also true that all possible input impedance loci could be plotted within the same circle. Such a plot is called the Smith chart after its inventor P. M. Smith, and the equations for the impedance loci will now be determined.

Figure 1 shows a typical reflection coefficient plot wherein the reflection coefficient is most conveniently plotted in polar coordinates. For reasons that will become apparent, the impedance loci are most conveniently plotted in rectangular coordinates so for this reason a rectangular coordinate axis system, u+jv, has been superimposed on the polar chart.

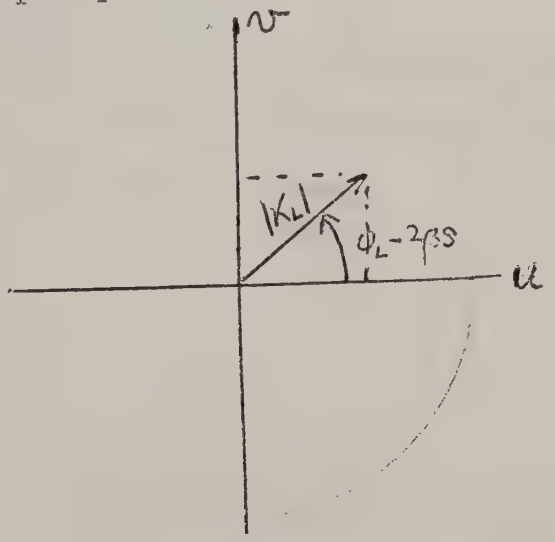



Fig. 1. Plot of K(S).

For the purpose of producing a general construction, one that will be useful for any line, it is necessary to normalize all impedances to the characteristic resistance. That is, the so-called "reduced impedance" or "per unit impedance" is defined as

$$Z' = Z/R_0 \quad . \tag{2}$$

Then using (2) in (1) the normalized impedance becomes

$$Z'(S) = \frac{1 + K(S)}{1 - K(S)} \tag{3}$$

which when solved for K(S) yields

$$K(S) = \frac{Z'(S) - 1}{Z'(S) + 1} \quad . \tag{4}$$

We may now express both sides of (4) in rectangular coordinates which gives

$$u + jv = \frac{r' + jx' - 1}{r' + jx' + 1} \quad . \tag{5}$$

If the real and imaginary parts of (5) are equated the resulting two equations are

$$u = \frac{r'^2 - 1 + x'^2}{(r'+1)^2 + x'^2} \tag{6}$$

$$v = \frac{2x'}{(r'+1)^2 + x'^2} \quad . \tag{7}$$

Since we seek impedance loci expressed in the u+jv coordinate system, we eliminate first x′ between Eqs. 6 and 7 to obtain an equation for the reduced resistance loci. After completing the square this equation becomes

$$\left[u - \frac{r'}{r' + 1}\right]^2 + v^2 = \left(\frac{1}{r'+1}\right)^2 \quad . \tag{8}$$

Similar manipulations to eliminate r′ between (6) and (7) give

$$(u-1)^2 + \left(v - \frac{1}{x'}\right)^2 = \left(\frac{1}{x'}\right)^2 \quad . \tag{9}$$

We shall now examine Eqs. 8 and 9 in some detail. Equation 8 will be recognized as the equation of a family of circles with centers

$$u = \frac{r'}{r' + 1} \quad , \qquad v = 0 \tag{10}$$

and radii

$$\text{radii} = \frac{1}{r' + 1} \quad . \tag{11}$$

Since r′ is bounded by

$$0 \leq r \leq \infty$$

it follows from (11) that

$$0 \leq \text{radii} \leq 1 \quad . \tag{12}$$

Let us determine the constant resistance loci for several examples from Eqs. 10 and 11. For the following four values of r′ the data are calculated and the results plotted in Fig. 2. One notes that all constant resistance circles lie within the unity radius circle of Fig. 1. One should also note that the circumference of the unity radius circle corresponds to r′ = 0, a point that was observed earlier when it was determined that this particular circle must correspond to pure reactance.

| r′ | radius | center |
|---|---|---|
| 0 | 1 | u = 0<br>v = 0 |
| 1 | 1/2 | u = 1/2<br>v = 0 |
| 1/2 | 2/3 | u = 1/3<br>v = 0 |
| ∞ | 0 | u = 1<br>v = 0 |

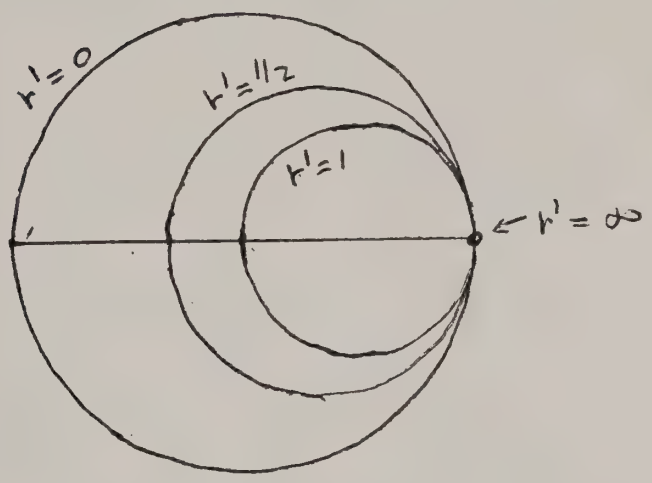



Fig. 2

Let us now examine Eq. 9 which will also be recognized as the equation of a family of circles. For the constant reactance family the circles have centers

$$u = +1 , \qquad v = 1/x' \tag{13}$$

and radii

$$\text{radii} = 1/x' \quad . \tag{14}$$

Since x′ is bounded by

$$-\infty \leq x' \leq \infty$$

the radii take on values

$$0 \leq \text{radii} \leq \infty$$

while the centers move along the line u = +1 from v = ∞ to v = -∞. The data for several examples are presented in the table and plotted in Fig. 3. It should be noted from Fig. 3 that the constant reactance circles of Eq. 9 do not lie entirely within the unity radius circle, however, only that portion of each circle that does lie within this circle need be plotted since it has been determined that all possible impedances on the line correspond to reflection coefficients which must lie within the unity radius circle. It should further be noted that the horizontal axis corresponds to

| x′ | radius | center |
|---|---|---|
| $-\infty$ | 0 | u = 1<br>v = 0 |
| -1 | 1 | u = 1<br>v = -1 |
| 0 | $\infty$ | u = 1<br>v = $\infty$ |
| +1 | 1 | u = 1<br>v = +1 |
| $+\infty$ | 0 | u = 1<br>v = 0 |

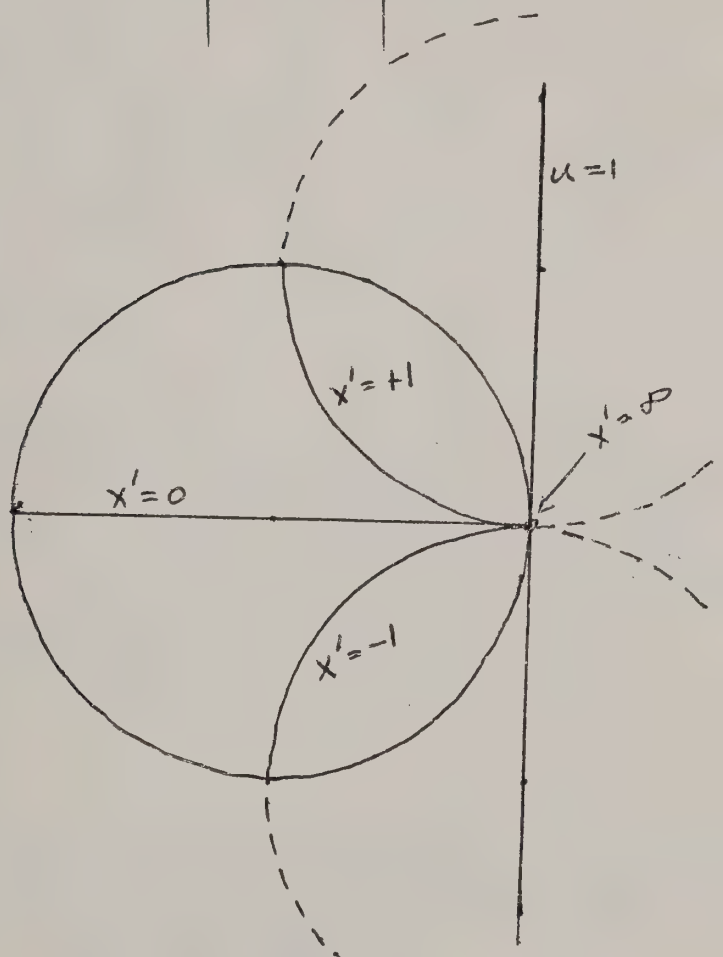



Fig. 3

x′ = 0 as we have seen previously. In addition the upper half of the plot includes all positive reactances while the lower half includes all negative reactances. By superimposing Figs. 2 and 3 one also notes, as was previously pointed out, that the left hand end of the horizontal axis is the point Z′ = r′ + jx′ = 0 while the right hand end is Z′ = r′ + jx′ = $\infty$.

B. Input Impedance Example

As a typical example consider use of the chart to determine input impedance. One should always keep in mind that the chart is just a graphical display of the transformation given by Eq. 3. That is, a superposition of a graph of K(S) onto the graph of $Z_{IN}(S)$. Thus the direct relationship of K(S) to position on a physical transmission line given by the crank diagram links every point on the Smith chart to distance from the load on the actual line. Given the line shown in Fig. 4 we seek the input impedance at $S_1$ = 45 cm. For this problem the logical starting point is the load, hence we calculate

$$K_L = \frac{Z_L - R_0}{Z_L + R_0}$$

$$= \frac{100 - j\,100 - 50}{100 + j\,100 + 50}$$

$$= 0.626/\underline{-29.6°} \quad .$$

Then

$$|K_L| = 0.626$$

$$\phi_L = -29.6° \quad . \tag{15}$$

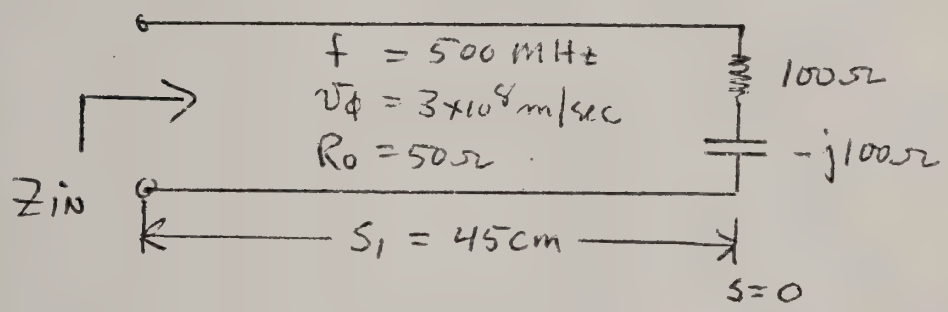



Fig. 4

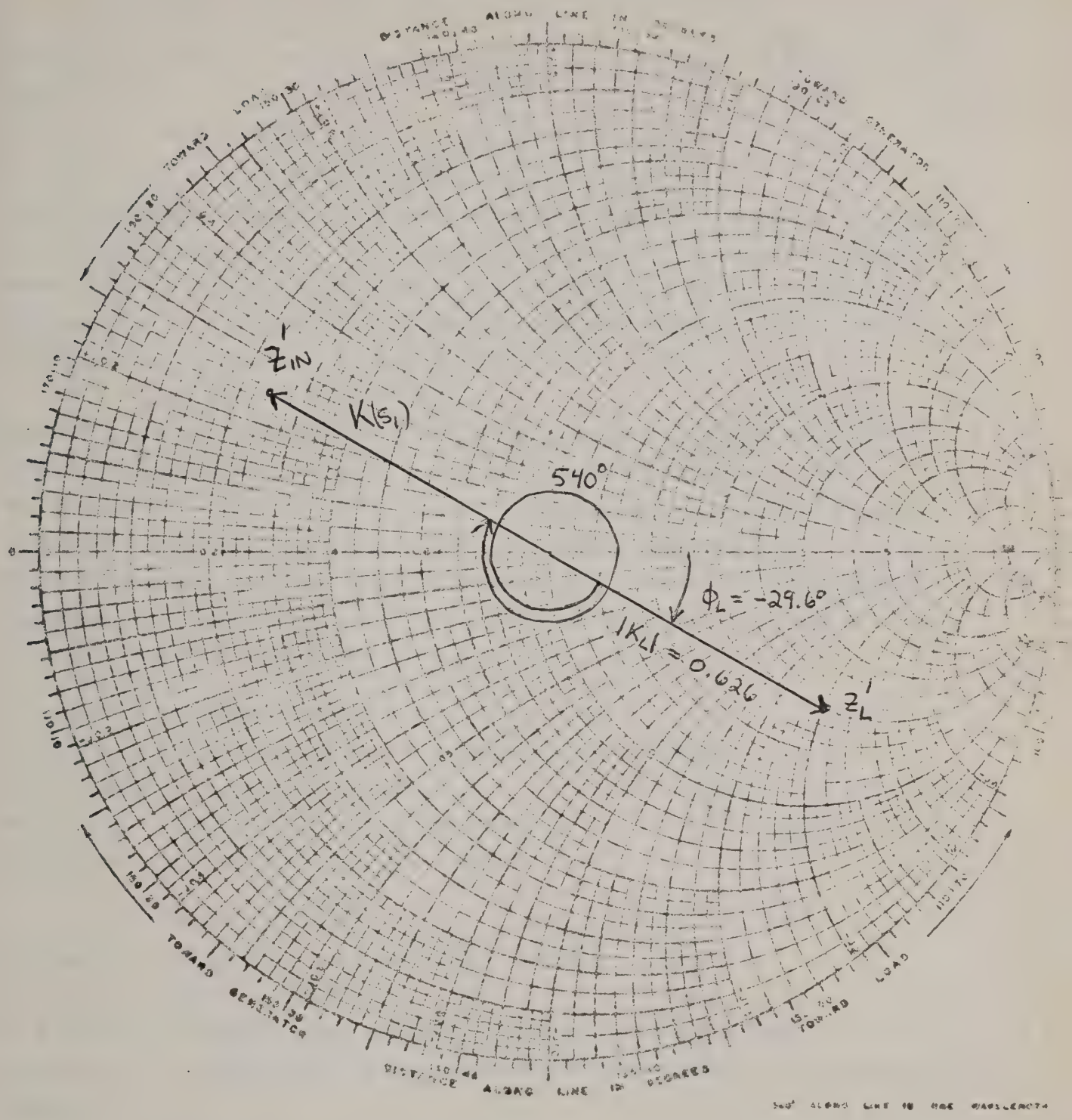
The vector given by (15) is plotted in Fig. 5 and thus $K_L$ has been located. One should note that Eq. 3 specialized to the load (S=0) yields

$$Z'_{IN}(S=0) = Z'_L = \left[\frac{1 + K_L}{1 - K_L}\right] \quad . \tag{16}$$

Since the Smith chart also performs the transformation given by (16) the given load impedance must be directly linked to $K_L$. Indeed in Fig. 5 one notes that the plotted point corresponds to Z' = 2 - j 2 which is the reduced load impedance. Hence when the load impedance (or any other impedance) is given, there is no need to perform the calculation of Eq. 15 since it is most easily accomplished graphically.

# REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE




MAG. / MIN IN DECIBELS

MAGNITUDE OF REFLECTION COEFFICIENT

TRANSITION LOSS IN DECIBELS

ATTENUATION SCALE — DECIBELS

To find the input impedance we now must locate the vector $K(S_1)$ where

$$K(S_1) = |K_L|\, e^{j\phi_L}\, e^{-2j\beta S_1} \quad . \tag{17}$$

That is, add negative angle $2\beta S_1$ to the angle of $K_L$. For this example $\lambda = 60$ cm, so

$$2\beta S_1 = 2\,\frac{2\pi}{60}\,45$$

$$= 3\pi \text{ radians; } 540^\circ \quad .$$

Thus, moving away from the load (toward the generator) a distance of 45 cm or, as is common practice to state, a distance of $\beta S_1 = 270$ "electrical degrees" corresponds to a rotation of the reflection coefficient vector through a clockwise angle of 540 polar degrees. This point is also plotted in Fig. 5 and the corresponding $Z'_{IN}$ is seen to be

$$Z'_{IN}(S_1) = 0.25 + j\ 0.25 \quad .$$

Hence for the 50 Ω line the input impedance is 12.5 + j 12.5 Ω. One should note that the reverse transformation is also possible and often times very useful in practice. That is, given the input impedance, what is the load impedance? Obviously all that is necessary is to locate the reduced input impedance and hence its corresponding reflection coefficient, rotate through a counterclockwise angle corresponding to the line length (2βS polar degrees) and locate thereby the reduced load impedance.

C. Inverting Complex Numbers

It is often required in transmission line calculations as well as in other completely unrelated applications to obtain the inverse of a complex number. This inversion is quickly accomplished on the Smith chart and the reason for this can be seen by inspection of Eq. 3:

$$Z'_{IN}(S) = \frac{1 + K(S)}{1 - K(S)} \quad . \tag{3}$$

One notes that since K(S) is a vector on this chart, then

$$K(S)\, e^{j\pi} = -K(S) \quad , \tag{18}$$

so a rotation on the chart of 180° polar locates the negative of the reflection coefficient. Now inspection of Eq. 3 reveals that this 180° rotation also corresponds to

$$Z'_{IN}(S+180^\circ) = \frac{1 - K(S)}{1 + K(S)} = \frac{1}{Z'_{IN}(S)} \quad . \tag{19}$$

Hence, to invert a complex number, one plots it on the chart and then rotates the vector corresponding to the point through 180 polar degrees. The new point so located will be the complex number inverse. This simple procedure provides

a wide variety of possibilities to aid in transmission line calculations because it allows one to quickly convert from reduced impedance to reduced admittance. Thus systems with parallel elements are easily handled. In fact the Smith impedance chart is also the Smith admittance chart since a 180° rotation accomplished the transformation.

When one works with admittance it should be noted that the normalization procedure is different, although of course, related to the impedance normalization. Since

$$Z' = Z/R_0$$

then

$$Y' = \frac{1}{Z'} = \frac{R_0}{Z}$$

$$= R_0 Y \tag{20}$$

$$= \frac{Y}{G_0} \quad .$$

D. Shunt Element Example

Consider the problem of determining the input impedance of the line shown in Fig. 6. We now have a shunt resistor at 12 cm from the load and seek the input impedance at 6 cm beyond the shunt element. This problem and many like it are easily handled by the graphical technique with the added feature of simple conversion between impedance and admittance.

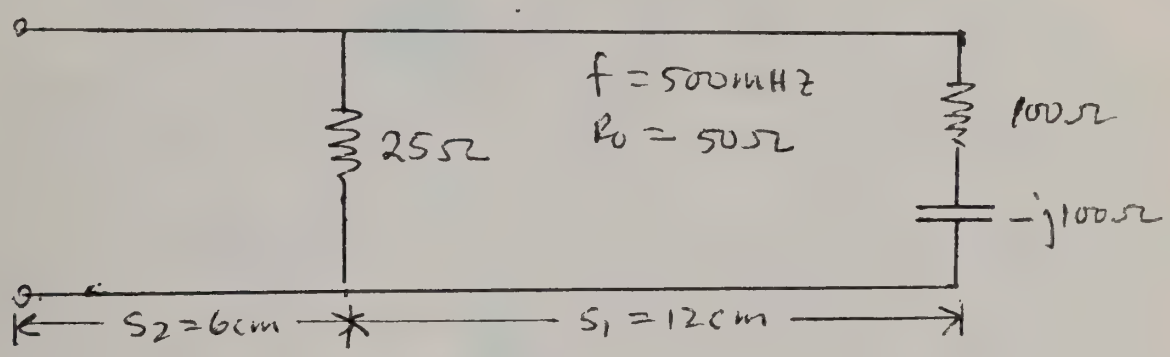



Fig. 6

We begin at the load by plotting the reduced load impedance, 2 - j 2, in Fig. 7. This $Z'_L$ is now converted to the $Z'_{IN}$ at 12 cm by a clockwise rotation through $2\beta S = 2(360°/60)\ 12 = 144$ polar degrees. This yields a reduced input impedance just to the right of the 25 Ω resistor of 0.23 - j 0.06. Because we now need to combine this impedance with that of the resistor it is convenient to convert to admittance. Through the 180° rotation shown we find the $Y'_{IN}(S_1)$ to be 4.1 + j 0.8. We now determine the conductance of the resistor to be 1/25 mho and through Eq. 20, G′ = 2. Hence the combination of the resistor and the line to the right yields a total reduced admittance of 6.1 + j 0.8. This point is plotted in Fig. 7 and now becomes the effective reduced load admittance for the 6 cm line. This addition is done graphically by moving along a constant susceptance circle. Note that the reflection coefficient magnitude and hence the VSWR is <u>different</u> on the two line sections.

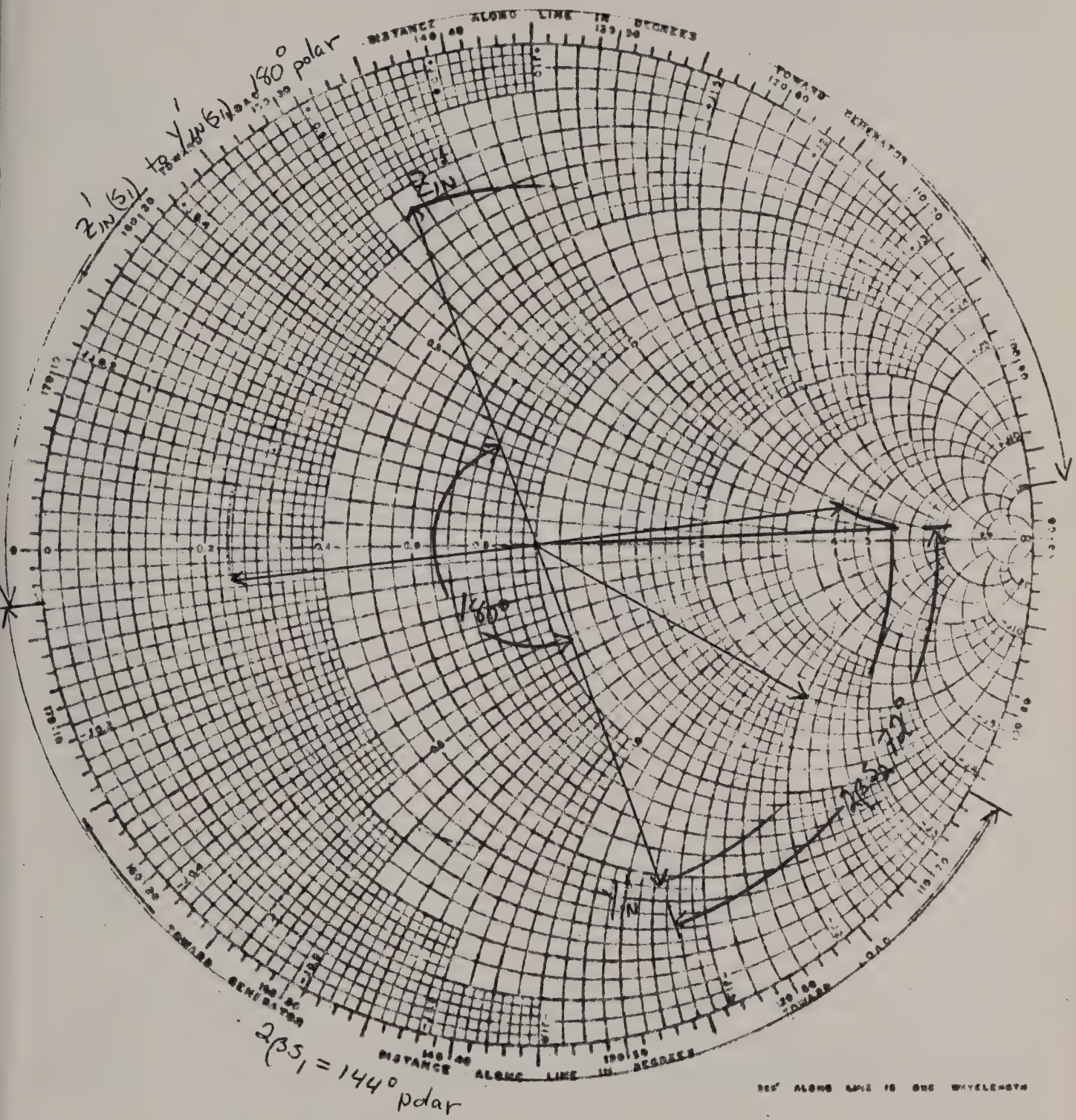
We proceed with the second part of the problem by transforming this new load admittance to the input by a rotation through $2\beta S_2 = 72$ polar degrees. This results in a reduced input admittance of 0.5 - j 1.3. Since it is the input impedance that we seek, a 180° rotation produces 0.25 + j .68. Finally the required input impedance is 50(0.25+j 0.68) = 12.5 + j 34 Ω.

E. Determination of Unknown Load Impedance

We have already considered the problem of determining impedance from line measurement data by calculation. Use of the Smith chart now makes those

# REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE




MAX. / MIN. IN DECIBELS

MAX. / MIN. IN DECIBELS

MAGNITUDE OF REFLECTION COEFFICIENT

V MAX. / V MIN. AND I MAX. / I MIN.

TRANSITION LOSS IN DECIBELS

ATTENUATION SCALE — DECIBELS

calculations unnecessary. The problem posed earlier of determining $Z_L$ from measurement of VSWR and $S_{MIN}$ is reproduced in Fig. 8.

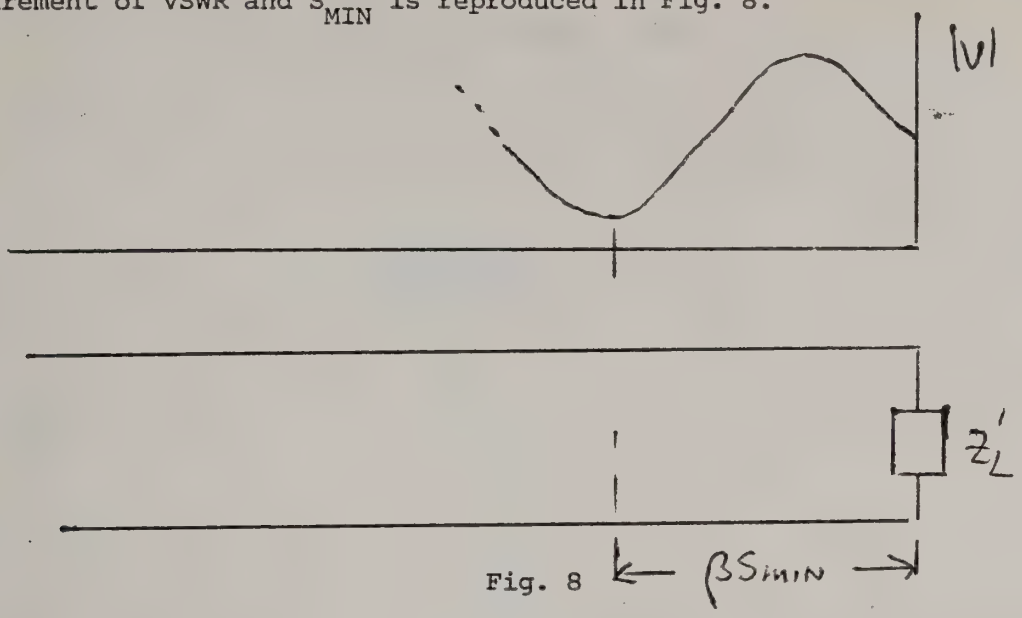



Fig. 8

We digress briefly to develop two important relationships between VSWR and impedance. Recalling Eq. 1 for input impedance in the form

$$Z_{IN} = R_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \tag{1}$$

and remembering from the crank diagram that $\phi - 2\beta S = 0$ at a voltage maximum and $\phi - 2\beta S = -\pi$ at a voltage minimum (Fig. 7, p. 14), we can then say that at a voltage maximum

$$Z_{IN_{MAX}} = R_0\left(\frac{1 + |K_L|}{1 - |K_L|}\right) , \tag{21}$$

while at a voltage minimum

$$Z_{IN_{MIN}} = R_0\left(\frac{1 - |K_L|}{1 + |K_L|}\right) . \tag{22}$$

Ths use of Eq. 78, p. 15, then converts Eqs. 21 and 22 into

$$Z_{IN_{MAX}} = R_0 \text{ VSWR} \tag{23}$$

$$Z_{IN_{MIN}} = R_0/\text{VSWR} . \tag{24}$$

We have already seen that the impedance at a maximum and a minimum is always real, but Eqs. 23 and 24 show in addition that it is simply related to the VSWR. Note that the reduced impedance at a voltage maximum is then

$$Z'_{IN_{MAX}} = \text{VSWR} , \tag{25}$$

so that the reduced resistance values on the <u>right</u> hand horizontal diameter of the Smith chart may also be read directly as VSWR.

Now returning to the impedance measurement problem, let us bear in mind the fact that any line is a complex impedance transformer and that the transformation is given graphically by the Smith chart. Hence knowledge of the impedance at any point on the line is tantamount to a knowledge of the impedance everywhere, including the load. Since we now have a simple way of determining the impedance at a minimum ($Z'_{IN}$ = 1/VSWR) we can readily find the load impedance if the distance from the minimum to the load is known. A minimum is chosen in practice rather than a maximum since the minima are narrower and greater precision is possible. The reduced input resistance at a minimum will always plot on the left hand horizontal diameter and then a rotation through $2\beta S_{MIN}$ polar degrees in a counterclockwise direction will locate the reduced unknown load impedance.

F. Unknown Load Example

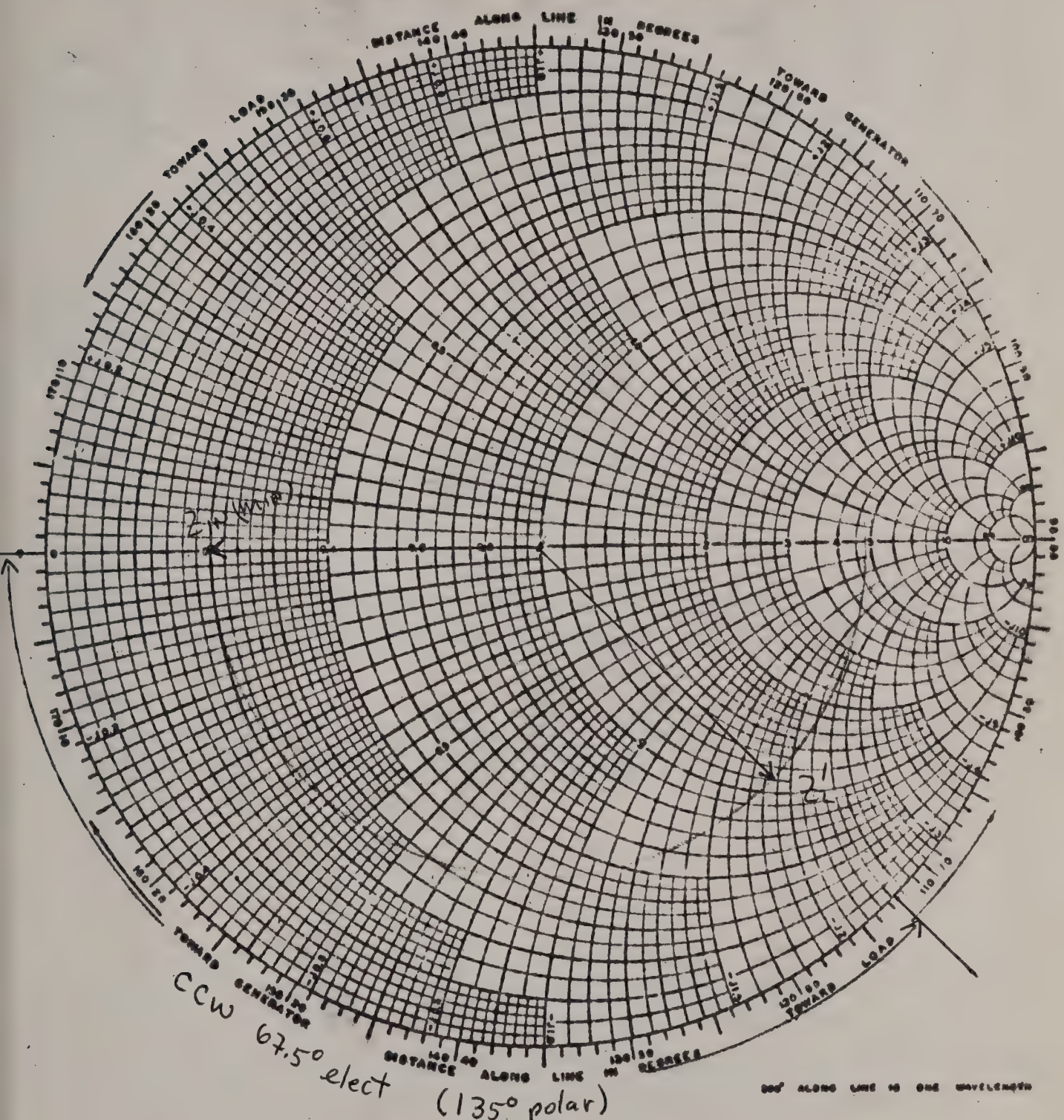
Suppose it is determined from measurement that the line VSWR is 5, and the distance from the first minimum to the load is 15 cm. In addition, take the wavelength to be 80 cm, and the characteristic resistance to be 75 Ω.

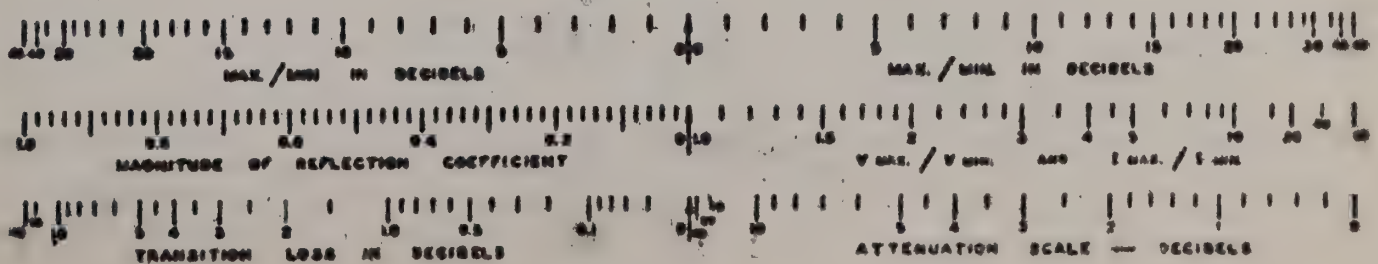
We begin by determining the reduced input resistance to be 1/5 = 0.2 and plot this point in Fig. 9 on the left horizontal diameter. The distance to the load now is $\beta S = (360/80)\ 15 = 67.5$ electrical degrees or $2\beta S = 135$ polar degrees. Rotation through this angle in a counterclockwise direction is shown on the graph yielding the reduced load impedance to be 1.1 = j 1.9. Hence the actual load must be

$$Z_L = 75(1.1 - j\ 1.9)$$
$$= 82.5 - j\ 142.5\ \Omega \quad .$$

# REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE





MAX. / MIN IN DECIBELS
MAX. / MIN. IN DECIBELS
MAGNITUDE OF REFLECTION COEFFICIENT
V MAX. / V MIN. AND I MAX. / I MIN.
TRANSITION LOSS IN DECIBELS
ATTENUATION SCALE — DECIBELS

## Transmission Lines as High Frequency Circuit Elements

### A. Introduction

A transmission line terminated in a pure reactance or an open or short circuit behaves as an energy storage device and can have impedance properties identical to those of lumped element resonant circuits. Consider, for example, the standing waves of voltage and current on a line short circuited at the load. Since the reflection coefficient of the load is $1\ e^{j\pi}$, the voltage becomes

$$\begin{aligned} V &= V_1\ e^{j\beta S} + V_2\ e^{-j\beta S} \\ &= V_1(e^{j\beta S} - e^{-j\beta S}) \\ &= j\ 2V_1\ \sin\beta S \quad . \end{aligned} \tag{1}$$

Under these conditions the line current becomes

$$\begin{aligned} I &= \frac{V_1}{R_0}\ (e^{j\beta S} + e^{-j\beta S}) \\ &= 2\ V_1/R_0\ \cos\beta S \quad . \end{aligned} \tag{2}$$

Equations 1 and 2 expressed in the time domain become

$$v(S,t) = 2\sqrt{2}\ |V_1|\ \sin\beta S\ \cos\left(\omega t + \frac{\pi}{2} + \theta_1\right) \tag{3}$$

$$i(S,t) = \frac{2\sqrt{2}\ |V_1|}{R_0}\ \cos\beta S\ \cos(\omega t + \theta_1) \quad . \tag{4}$$

One sees from Eqs. 3 and 4 that the voltage and current are 90° out of <u>time phase</u> as required for an ideally infinite Q resonator.

### B. Quarter Wavelength Line

Let us examine the input impedance of a transmission line shorted at the load end. For such a line the input impedance is given by

$$\begin{aligned} Z_{IN} &= R_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \\ &= R_0\left[\frac{1 - e^{-j2\beta S}}{1 + e^{-j2\beta S}}\right] \\ &= j\ R_0\ \tan\beta S \quad . \end{aligned} \tag{5}$$

At the frequency, $f_0$, for which the line is just a quarter wavelength long, $\beta S = \pi/2$, Eq. 5 (or the Smith chart) yields the value infinity. Consider, now excitation of the same line at frequencies somewhat <u>below</u> $f_0$. Now the wavelength has increased and the line is now somewhat <u>shorter</u> than $\lambda/4$. That is, $\beta S < \pi/2$ and Eq. 5 (or the Smith chart) yields large values of $X_L$ for the input impedance. If now the excitation frequency is increased above $f_0$, the line becomes <u>longer</u> than $\lambda/4$ and the above equation gives large values of $X_C$ for $Z_{IN}$. This behavior is identical to that of a parallel LC circuit in the close vicinity of resonance.

If now the line section is open circuited, considerations similar to those above lead one to the expression for the input impedance as

$$Z_{IN} = -j\, R_0 \cot\beta S \quad . \tag{6}$$

This expression (or the Smith chart) gives zero for the $Z_{IN}$ at $f_0$, and below $f_0$ $Z_{IN}$ takes on small values of capacitive reactance while above $f_0$ small values of $X_L$ are found. This is easily seen from (6) by writing

$$Z_{IN} = -j\, R_0 \cot\left(\frac{\pi}{2}\frac{f}{f_0}\right) \tag{7}$$

where the substitutions

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_\phi} \quad , \qquad S = \frac{\lambda_0}{4} = \frac{v_\phi}{4f_0}$$

have been used. One attributes this type of behavior to a series LC circuit and hence the terminals of the open quarter wavelength line are equivalent to those of a series LC circuit in the proximity of resonance.

Quarter wavelength lines are also used for matching resistive loads. For an arbitrary load the quarter wavelength input impedance becomes

$$\begin{aligned} Z_{IN} &= R_0\left[\frac{1 + K(S)}{1 - K(S)}\right] \\ &= R_0\left[\frac{1 - K_L}{1 + K_L}\right] \\ &= R_0/Z_L' = R_0^2/Z_L \quad . \end{aligned} \tag{8}$$

If the load, $Z_L$, is real, then the input impedance is also real and is

$$R_{IN} = R_0^2/R_L \quad . \tag{9}$$

Equation (9) allows one to convert an arbitrary load resistance into the characteristic resistance of a driving line by properly choosing the characteristic resistance of a quarter wavelength matching section. Fig. 1 shows such a system in which $R_0 = \sqrt{R_{01}R_L}$.

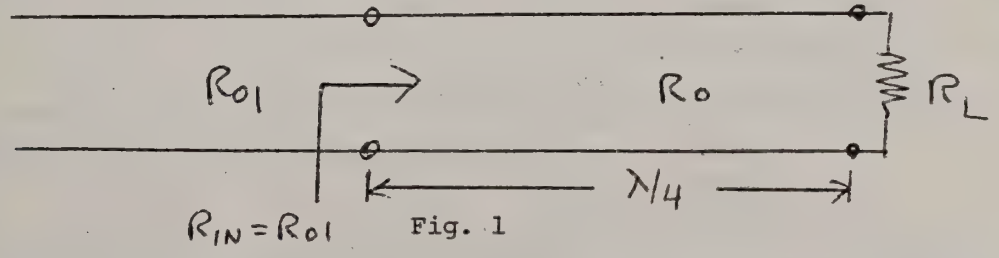



Fig. 1

Quarter wavelength lines are also used as supports for transmission lines which in turn are only used over a narrow frequency range. Since Eq. 5 yields an infinite input impedance the quarter wavelength section is an open circuit at its terminals and can be used, as depicted in Fig. 2, as a very stable mechanical support.

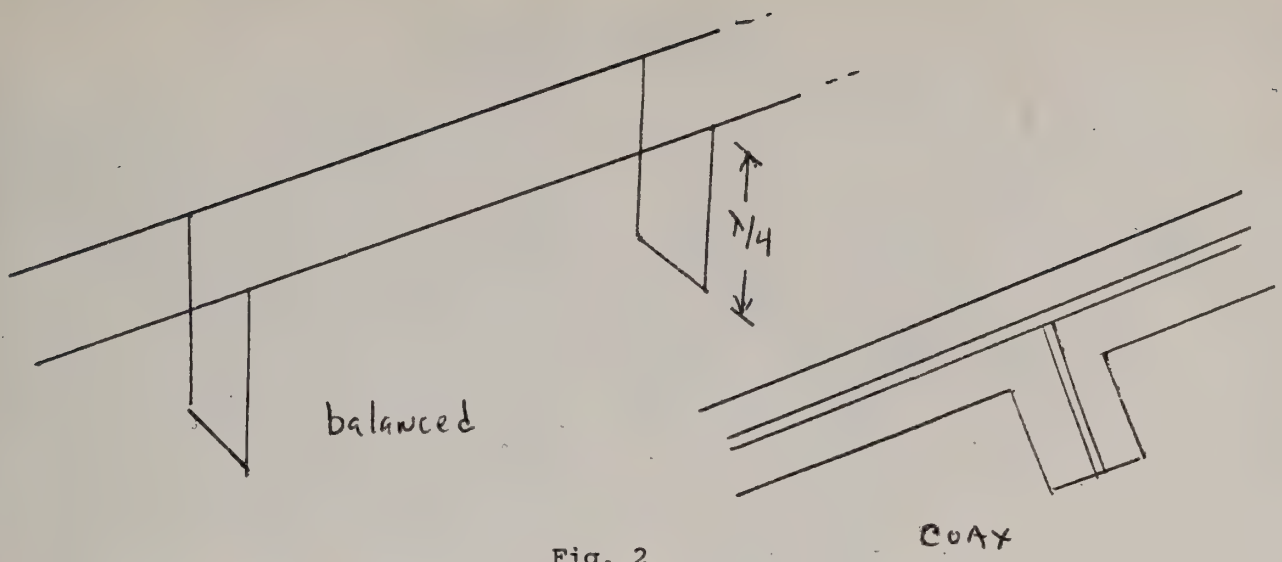



Fig. 2

C. Half Wavelength Lines

For line length in the vicinity of $\lambda/2$, Eq. 6 may be written, using the technique employed in Eq. 7, as

$$Z_{IN} = -j\ R_0\ \cot(\pi f/f_0)\ , \tag{10}$$

i.e., here $S = \lambda_0/2 = v_\phi/2f_0$. Hence Eq. 10 describes the input impedance of an open circuited line near the frequency $f_0$ for which the line is one half wavelength long. Recalling the behavior of cotangent near $\pi$ one sees that this line is equivalent to a shorted quarter wavelength line. In a like manner, it is easy to see that an open quarter wavelength line is identical in behavior to a shorted half wavelength line.

The possible resonator configurations described above allow the high frequency circuit designer considerable freedom in choosing the right circuit for a given application. In addition, since open circuited lines are not desirable since the open end has fringing fields which tend to couple to other circuits, these lines are somewhat foreshortened and terminated in capacitance. If this capacitance is made variable then an additional feature results, in that the electrical line length becomes adjustable.

D. Filter Applications

As an example of the use of low loss transmission line elements with capacitive loading, the following excerpt from Hewlett-Packard Journal, February 1974, is included on the next page. One should note that for such filters the choice of $R_0$ is important. The impedance level attainable in the antiresonance condition (parallel resonance) is directly proportional to $R_0$.

# Laboratory Notebook

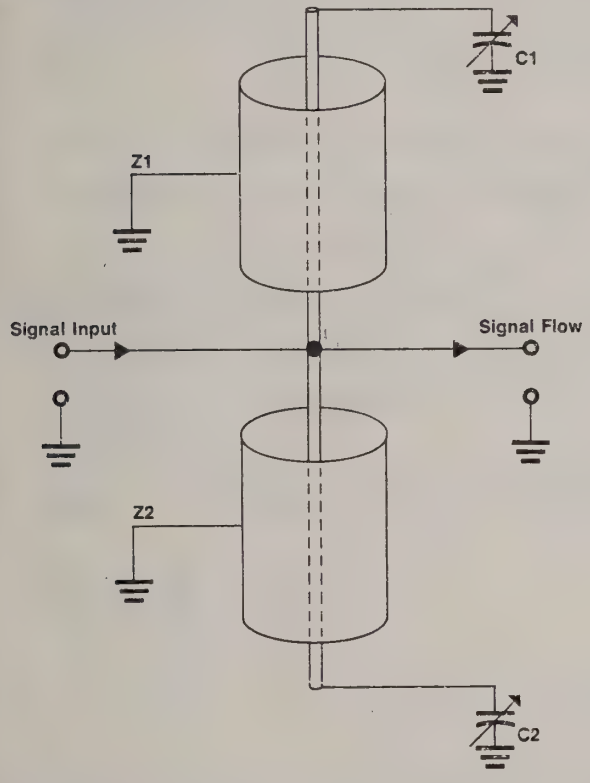
## Sharp Cut-off Filters for That Awkward UHF Band

*With discrete components too lossy, and cavities too bulky, the design of compact, sharp cut-off filters for the frequencies between 200 and 1000 MHz poses a problem.*

*In designing the Model 78101A ECG Telemetry Receiver,* we faced this problem. Here was needed a filter that would remove an image frequency at 423 MHz without overly attenuating the carrier at 467 MHz (IF of 22 MHz and LO at 445 MHz). Ordinarily this would be done with bandpass filters, several in series being required to get adequate rejection in this frequency band. Several filters in series, though, means considerable loss.*

*This problem was solved by using a transmission-line filter. This filter is not only effective, but it is also inexpensive.*

*Shown in the drawing below, the filter has two transmission-line sections that form a series-resonant trap for the image frequency and a parallel-resonant passband for the carrier. Operation is as follows: Capacitor C1 tunes transmission-line Z1 to series resonance at 423 MHz, thus effectively shorting out any*



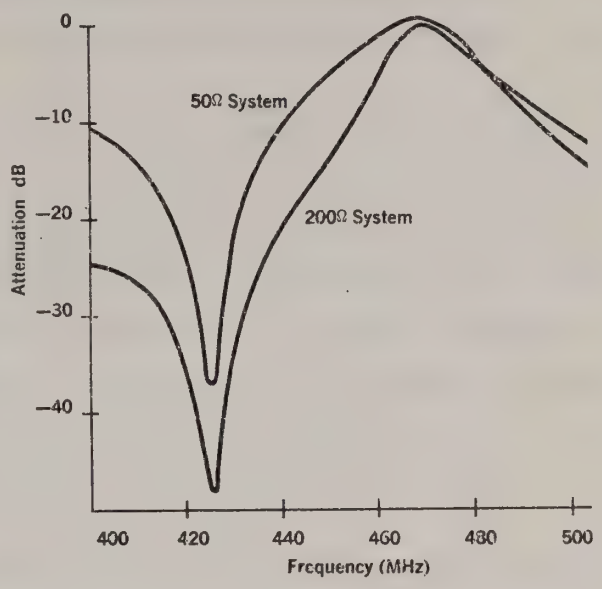

*signals at the image frequency. At higher frequencies, Z1 is inductive but transmission-line Z2, being shorter, is capacitive. C2 is tuned to resonate Z2 with Z1 at 467 MHz so the carrier 'sees' a parallel resonant circuit, and passes with negligible loss.*

*As shown in the plot of frequency response, this design easily achieves more than 35 dB rejection of signals at the image frequency in a 50Ω system, and more than 45 dB in a 200Ω system.*

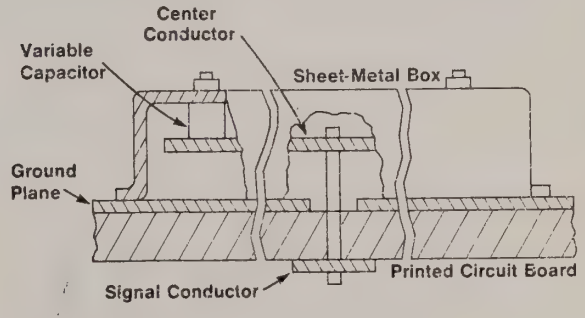
### Design Execution

*There are a number of ways that such a circuit can be constructed. Using lengths of coaxial cable was not entirely satisfactory, however, because the fringing fields at the open ends of the cable where the capacitors are attached interacted with other filters nearby.*

*Larsen et al, "An Effective ECG Telemetry System," Hewlett-Packard Journal, April 1972.




*The design that evolved uses a long, narrow, sheet-metal box with an open side that is clamped to the ground plane of a printed-circuit board, forming the outer conductor of a coaxial line. The center conductor is a flat bar mounted at each end by a trimmer capacitor, as shown in the drawing. This forms a completely shielded, low-loss, air-dielectric transmission line.*




*The single structure contains both sections of coaxial line. The signal lead, brought from the other side of the printed-circuit board through a hole in the ground plane, attaches to the center conductor at the point that divides the line into the series-resonant and parallel-resonant sections. Actually, two signal leads are used, one for the input to the filter and one for the output, eliminating a common impedance between input and output that would degrade performance. These leads attach on opposite sides of the center-conductor bar.*

*Two of these filters are used in the HP Model 78101A ECG Telemetry Receiver, one at the antenna input and one at the output of the two-stage tuned RF amplifier. Overall, an image frequency rejection of more than 100 dB is obtained.*


*James Larsen*
*Richard Dilman*
*Richard Tverdoch*

*Medical Electronics Division*

E. Practical Resonators

The preceding considerations are all based on ideal lossless lines and all resonators constructed from such ideal components would have infinite Q. Hence, when designing resonators from transmission lines even very small loss cannot be ignored.

To find the relationship of Q to line loss we begin by a consideration of the decay of energy in a transmission line resonator. Imagine such a resonator excited with a steady state source and, at t = 0, the source is disconnected. Oscillations will decay exponentially with time exactly as they built up exponentially when the source was first connected, a situation observed in several applications discussed previously. The decay of the terminal voltage of the resonator is plotted in Fig. 3. If we call the voltage at t = 0 (switch opening) V', then the voltage one oscillation period later is given by

$$V'' = V' e^{-\xi\tau} \tag{11}$$

where $\xi$ is the decay constant. The initial energy stored in the resonator is proportional to the initial terminal voltage squared, hence

$$W' = K |V'|^2 \quad . \tag{12}$$

The energy dissipated during the first cycle must be the difference between the energy given by (12) and that stored after a time $\tau$,

$$W_{diss/cycle} = K |V'|^2 - K |V''|^2 \tag{13}$$

or using (11)

$$W_{diss/cycle} = K |V'|^2 (1-e^{-2\xi\tau}) \quad . \tag{14}$$

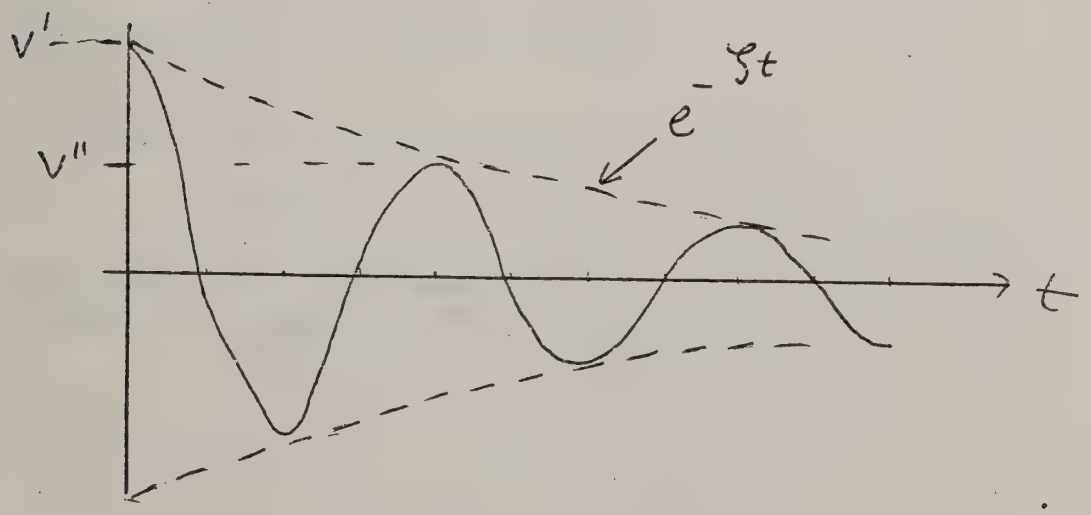



Fig. 3

Since the fundamental definition of quality factor is given by

$$Q = \frac{2\pi\, W_{\text{stored max}}}{W_{\text{diss/cycle}}} = \frac{\omega_0\, W_{\text{stored max}}}{P_{\text{diss}}} \quad , \tag{15}$$

we have, using (12) and (13),

$$Q = \frac{2\pi}{1 - e^{-2\xi\tau}} \quad . \tag{16}$$

For high Q (Q>10) and high frequencies

$$2\xi\tau \ll 1 \qquad \text{and} \qquad e^{-2\xi\tau} \cong 1 - 2\xi\tau \quad , \tag{17}$$

hence Eq. 16 becomes

$$Q \cong \frac{\pi}{\xi\tau} \quad . \tag{18}$$

For example, since now

$$V = V' e^{-\pi t/Q\tau} \quad , \tag{19}$$

a resonator at 100 MHz having a Q of 3000 would have its voltage decay to 37% of its initial value in about 10 μsec.

It is now a simple matter to relate the decay constant, $\xi$, to the attenuation constant, $\alpha$, of the line. Since the transmission line supports traveling waves, one can view the voltage decay either as a function of time at a fixed point or as a function distance at a fixed time. Since the waves travel one wavelength in one period, the voltage as a function of distance equation analogous to Eq. 11 is

$$\begin{aligned} V &= V' e^{-\alpha X} \\ &= V' e^{-\alpha\lambda} \quad . \end{aligned} \tag{20}$$

Hence comparing (11) and (20) one sees

$$\alpha\lambda = \xi\tau \quad . \tag{21}$$

Using (21) in (18) then gives

$$Q \cong \frac{\pi}{\alpha\lambda} \quad . \tag{22}$$

Hence, given the attenuation constant for a transmission line, the resultant Q of a resonator constructed from this line is determined.

Figure 4 shows the values of Q that can be obtained from a resonator constructed from coaxial line. The line is air insulated copper and the outer to inner radius ratio b/a = 3.6. This latter condition results in optimum Q as will be shown shortly. Notice that the Q values are far greater than those expected from lumped element resonators.

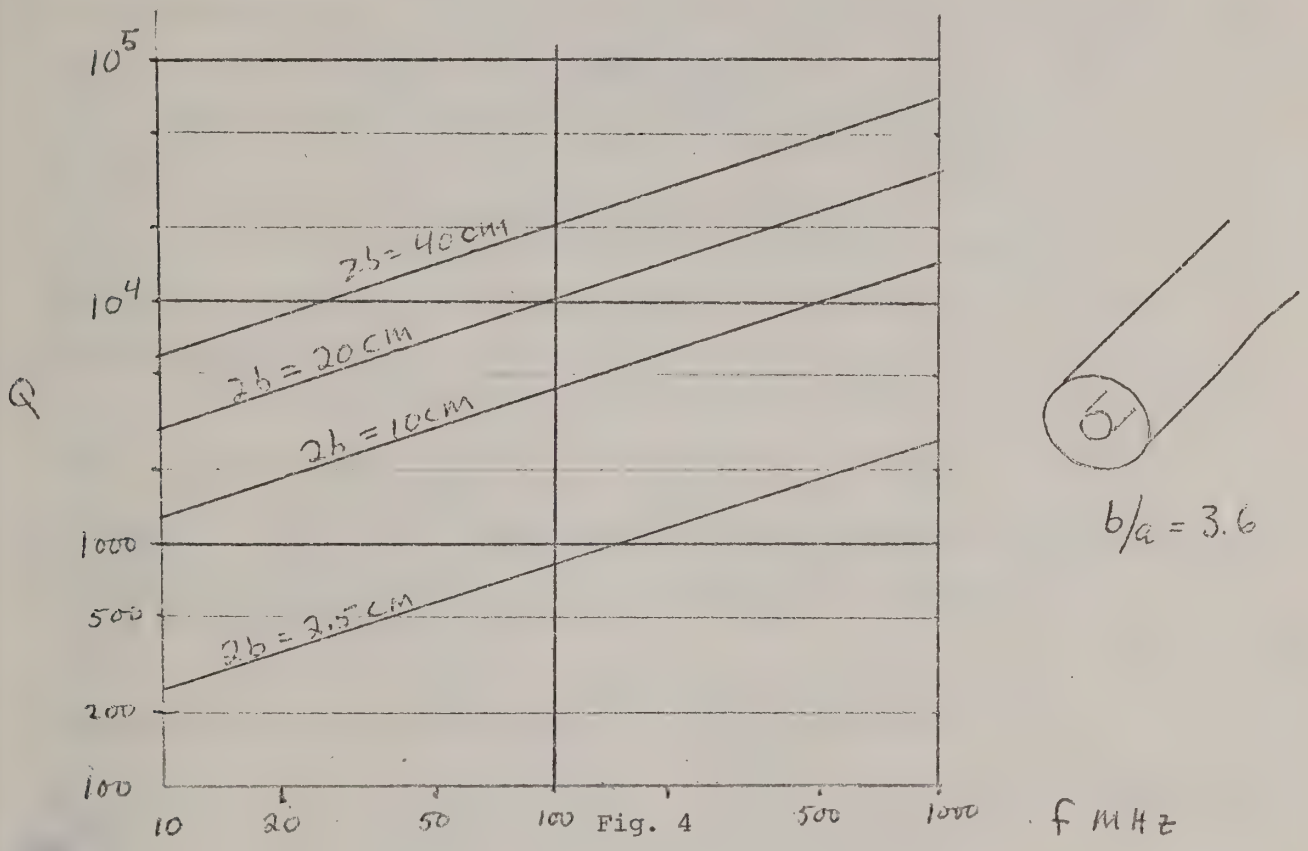



Fig. 4

F. Design Constraints

We shall now examine the condition under which optimum Q is obtained. Equation 22 indicates that this situation will result when the loss constant, $\alpha$, is a minimum. The loss constant is given by Eq. 23 for a coaxial line. (This equation will be derived later when skin effect is discussed.)

$$\alpha = \frac{R_s}{2\eta} \frac{\frac{1}{a} + \frac{1}{b}}{\ln \frac{b}{a}} \tag{23}$$

where

$R_s$ = surface resistance of metal

$\eta$ = intrinsic impedance of dielectric.

We wish to examine the behavior of $\alpha$ for the constraint of b = constant and $0 \leq a \leq b$. To this end we form

$$\frac{d\alpha}{da} = 0 = \frac{d}{da}\left[\frac{\frac{1}{a} + \frac{1}{b}}{\ln \frac{b}{a}}\right]. \tag{24}$$

Carrying out the differentiation results in

$$\frac{b}{a} = e^{1+(1/(b/a))} \quad . \tag{25}$$

The transcendental equation in (25) has the solution

$$\frac{b}{a} = 3.6 \quad . \tag{26}$$

Hence a coaxial line will always have minimum loss and resonators constructed from the line will have maximum Q when the ratio b/a = 3.6 is demanded. That such an optimum ratio should exist is easily justified physically. Consider the range of a. For small a, the center conductor will have high resistance and $\alpha$ will increase. For a approaching b, the line capacitance will increase and hence its characteristic resistance will decrease. This results in increased current and hence increased losses. Thus the loss will minimize somewhere between a = 0 and a = b, namely b/a = 3.6.

The minimum described above is rather broad and values of b/a in the range

$$2.3 \leq b/a < 6 \tag{27}$$

will result in values of $\alpha$ within 10% of the value obtained with b/a = 3.6. Since the characteristic resistance of coaxial line is given by

$$R_0 = \frac{\eta}{2\pi} \ell n \frac{b}{a} \, , \tag{28}$$

the value of 3.6 in Eq. 28 results in (for air insulation)

$$R_0 = 77 \ \Omega \quad . \tag{29}$$

Considering the range described by (27) one sees that lines with characteristic resistances in the vicinity of 77 $\Omega$ will have minimum loss.

Consider now another criterion which may dictate the choice of transmission line, that of power handling capability. This quantity is also limited by small and large radii since dielectric breakdown will occur in each limit. For coaxial line, the maximum electric field occurs at the center conductor and with E given by

$$E_r = \frac{\rho_\ell}{2\pi\epsilon r} \tag{30}$$

one determines

$$E_{r_{max}} = \frac{\rho_\ell}{2\pi\epsilon a} \quad . \tag{31}$$

Using (31) in (30) allows the line voltage to be written as

$$\begin{aligned} V &= \int_a^b E_r \, dr \\ &= \int_a^b E_{r_{max}} \frac{a}{r} \, dr \\ &= E_{r_{max}} \, a \, \ell n \frac{b}{a} \quad . \end{aligned} \tag{32}$$

We may now express the power in terms of physical line parameters by using (32) and (28) in

$$P = \frac{V^2}{R_0}$$

$$= \frac{E_{r_{max}}^2 \, a^2 \left(\ell n \, \frac{b}{a}\right)^2}{\frac{\eta}{2\pi} \, \ell n \, \frac{b}{a}}$$

$$= \frac{2\pi \, E_{r_{max}}^2}{\eta} \left[a^2 \ell n \, \frac{b}{a}\right] \quad . \tag{33}$$

To maximize the power for fixed b one forms

$$\frac{dP}{da} = 0 = \frac{d}{da} \left[a^2 \ell n \, \frac{b}{a}\right] \quad , \tag{34}$$

which results in

$$\ell n \, \frac{b}{a} = \frac{1}{2}$$

$$\frac{b}{a} = 1.65 \quad . \tag{35}$$

Hence, the maximum power condition does not coincide with the minimum loss condition.

Using Eq. 28 one sees that line constructed for maximum power will have a characteristic resistance of

$$R_0 = \frac{\eta}{2\pi} \ell n \; 1.65$$

$$= 30 \; \Omega \quad . \tag{36}$$

The commonly found line impedance of 50 Ω is a compromise between the maximum power and minimum loss conditions.

## Stub Matching

### A. Single Stub Matching

In many practical situations it becomes desirable to match a given load to a driving transmission line. In general, line losses are reduced as is driving point impedance variation with frequency. A simple method of matching using one shunt line element is called single stub matching.

Single stub matching can be easily understood by referring to Fig. 5. In view of the fact that a length of line is an impedance (or admittance) transformer, we could imagine a length of line, $S_1$, chosen such that the reduced load admittance, $Y_L'$, is transformed into a reduced input admittance equal to $1.0 \pm jB$. Then at this point a susceptance equal to $\mp jB$ could be placed in <u>shunt</u> with the line resulting in a combined reduced input admittance of

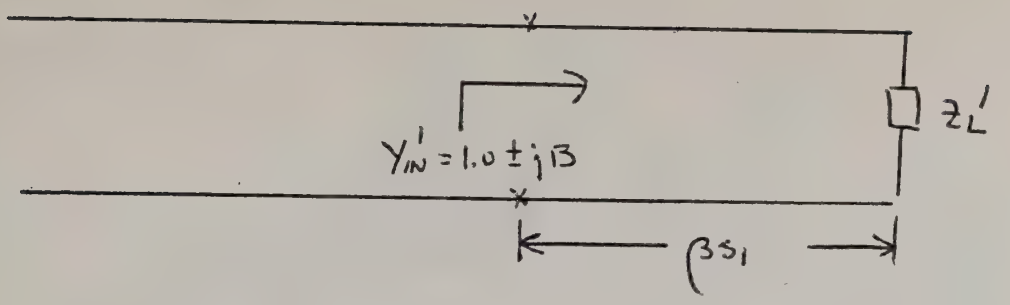



Fig. 5

$Y'_{IN} = G'_{IN} = 1.0$. Under these conditions the line is matched. The shunt susceptance, $\mp jB$, is conveniently obtained in practice from a variable length of of short-circuited line called a "stub."

The above considerations will be clarified by an example. The use of the Smith chart to accomplish the design of the line lengths greatly simplifies the problem. Suppose a load of $Z_L = 20 - j\ 55\ \Omega$ is placed on a 50 $\Omega$ line. The reduced load impedance is plotted in Fig. 6. Since the design problem is greatly simplified by working in the admittance plane, we convert $Z'_L$ into $Y'_L$ by a 180° polar degree rotation resulting in $Y'_L = 0.3 + j\ 0.8$. We now seek the line length $S_1$ that will transform $Y'_L = 0.3 + j\ 0.8$ into $Y'_{IN} = 1.0 \pm jB$. Clearly, on the Smith chart this requires an electrical length of 27.5 electrical degrees (toward the generator), thus $\beta S_1 = 27.5°$. At this position the input admittance is $1.0 + j\ 2.0$, hence the required susceptance for the shunting stub must be $-j\ 2.0$. To obtain such a susceptance from a short-circuited stub requires a length of $\beta S_2 = 26.5$ electrical degrees as shown in Fig. 6. The completed matching design is shown in Fig. 7. Note that the VSWR on line 1 is 5.8 while on line 2 it is $\infty$, however on the remainder of the system it is 1.

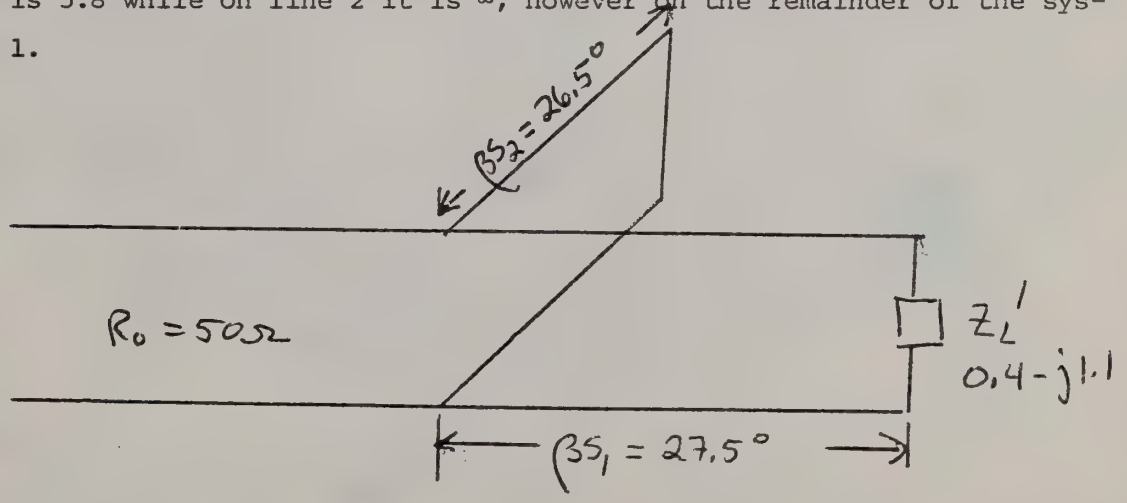
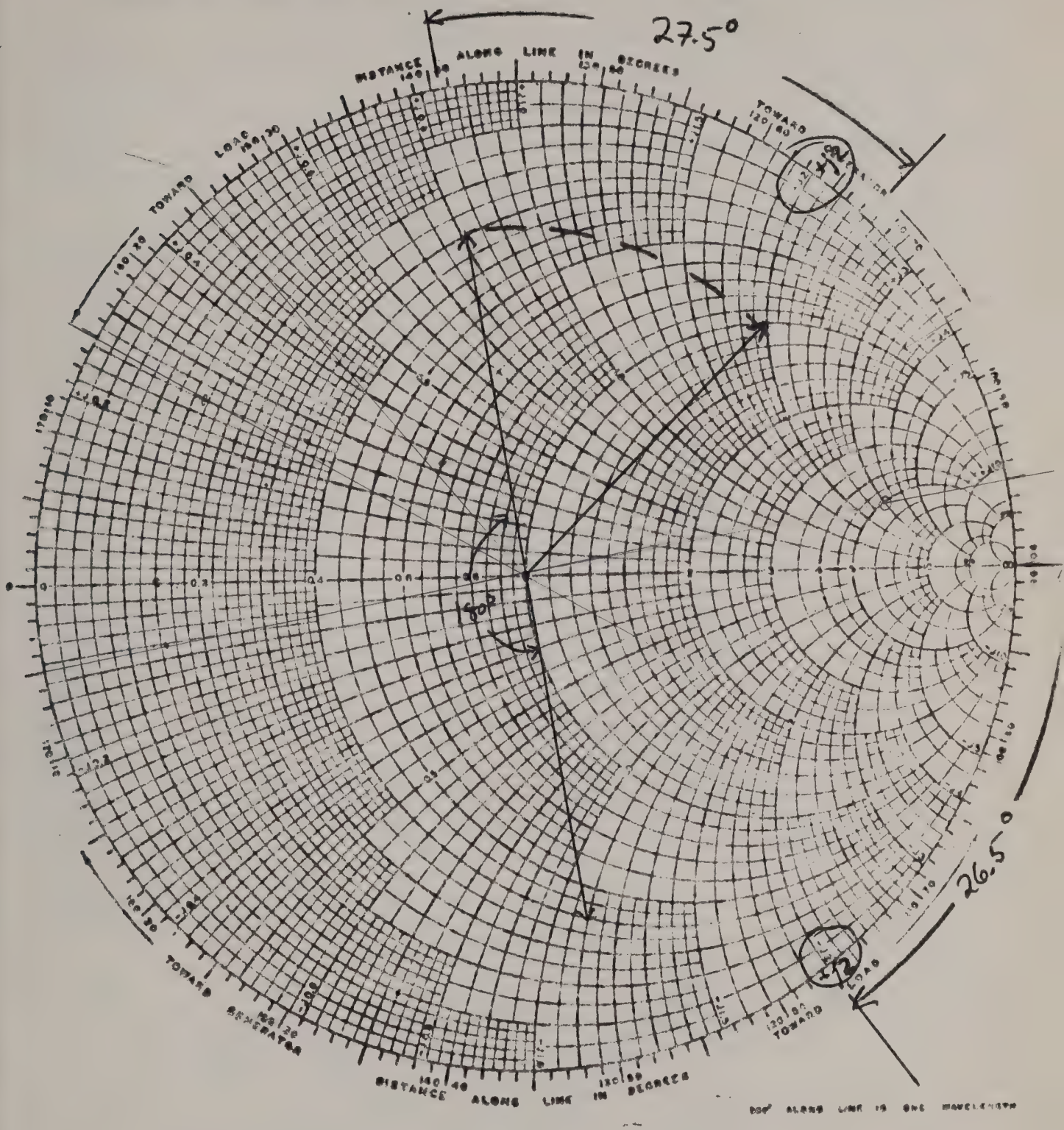
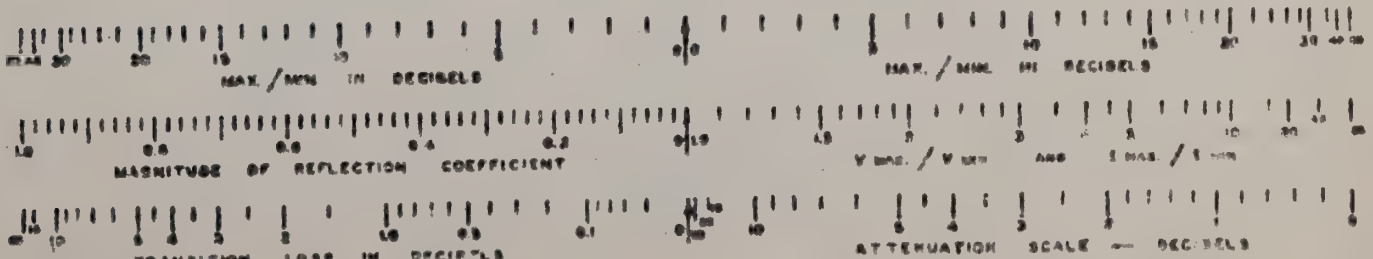



Fig. 7

# REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE





MAX. / MIN. IN DECIBELS
MAX. / MIN. IN DECIBELS
MAGNITUDE OF REFLECTION COEFFICIENT
TRANSITION LOSS IN DECIBELS
ATTENUATION SCALE — DECIBELS

B. Double Stub Matching

The single stub matching just described has the advantage that any load containing a real part can be matched to any line using the procedure. However, the method has a practical disadvantage in that repositioning of the stub location is difficult. A method which alleviates this difficulty is double stub matching. One pays a price using this scheme however, in that for a given spacing of the two stubs not all loads can be matched.

A double stub matching system is shown in Fig. 8. Here two short circuited stubs are connected across the line and spaced by $d_2$ meters. Understanding of

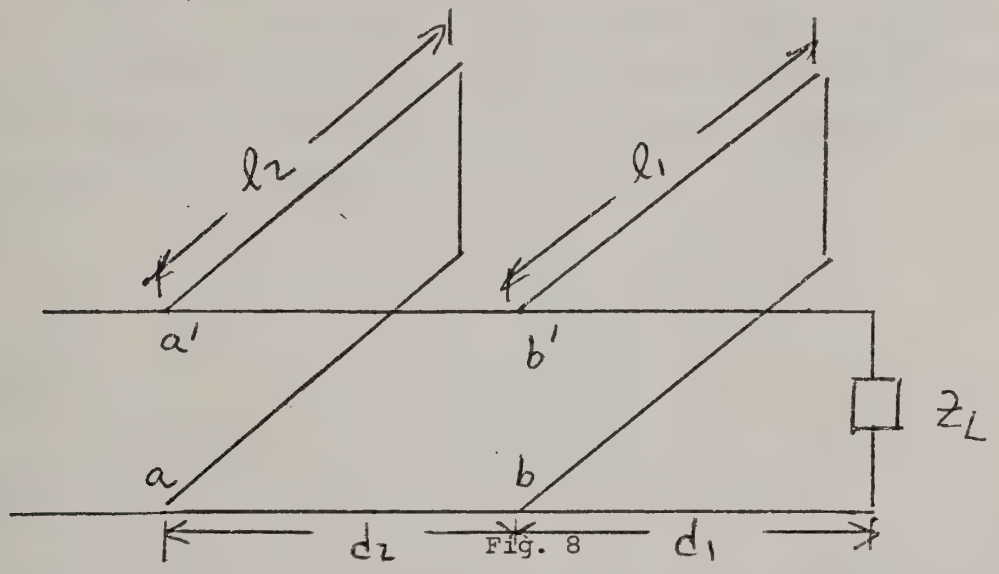


Fig. 8

the method is best gained by considering the reduced admittance that must be presented by the line at the terminals aa' <u>prior</u> to connecting stub number 2. That is, consider the input admittance found from Fig. 9. Making use of one's knowledge of single stub matching it is clear that $Y'_{IN_a} = 1.0 \pm jB'$ because stub 2 can only add $Y'_{IN_{S_2}} = \mp jB'$. That is, when stub 2 is connected, the sum of $Y'_{IN_a}$ and $Y'_{IN_{S_2}}$ will result in a match.

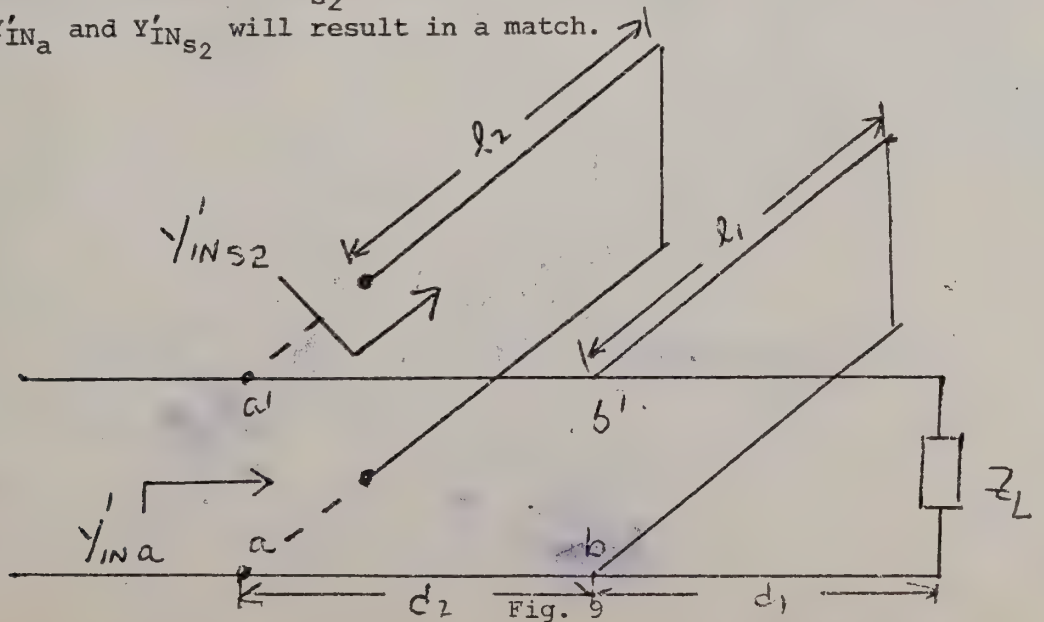


Fig. 9

Since the possible values of $Y'_{IN_a}$ are determined, the possible input admittance at terminals bb' is also determined because the transformation by line length $d_2$ is unique and readily obtainable from the Smith chart. That is, one asks the question, what admittance is required at terminals bb' in order that it be transformed by line length $d_2$ into the admittance $Y'_{IN_a} = 1.0 \pm jB$? This question is easily answered by referring to the Smith chart. The locus of all possible $Y'_{IN_a}$ is just the G'= 1 circle on the chart. Then in order to determine the required admittance at bb', $Y'_{IN_b}$, it is only necessary to transform each point on the G'= 1 circle back through $\beta d_2$ electrical degrees <u>toward the load</u>. That is, a rotation on the chart through $\beta \ell_2$ electrical or $2\beta d_2$ polar degrees. Such a transformation is shown in Fig. 10. Because the circular geometry will be preserved when each point is rotated through the same angle, the locus of possible $Y'_{IN_b}$ is called the "shifted G'= 1 circle."

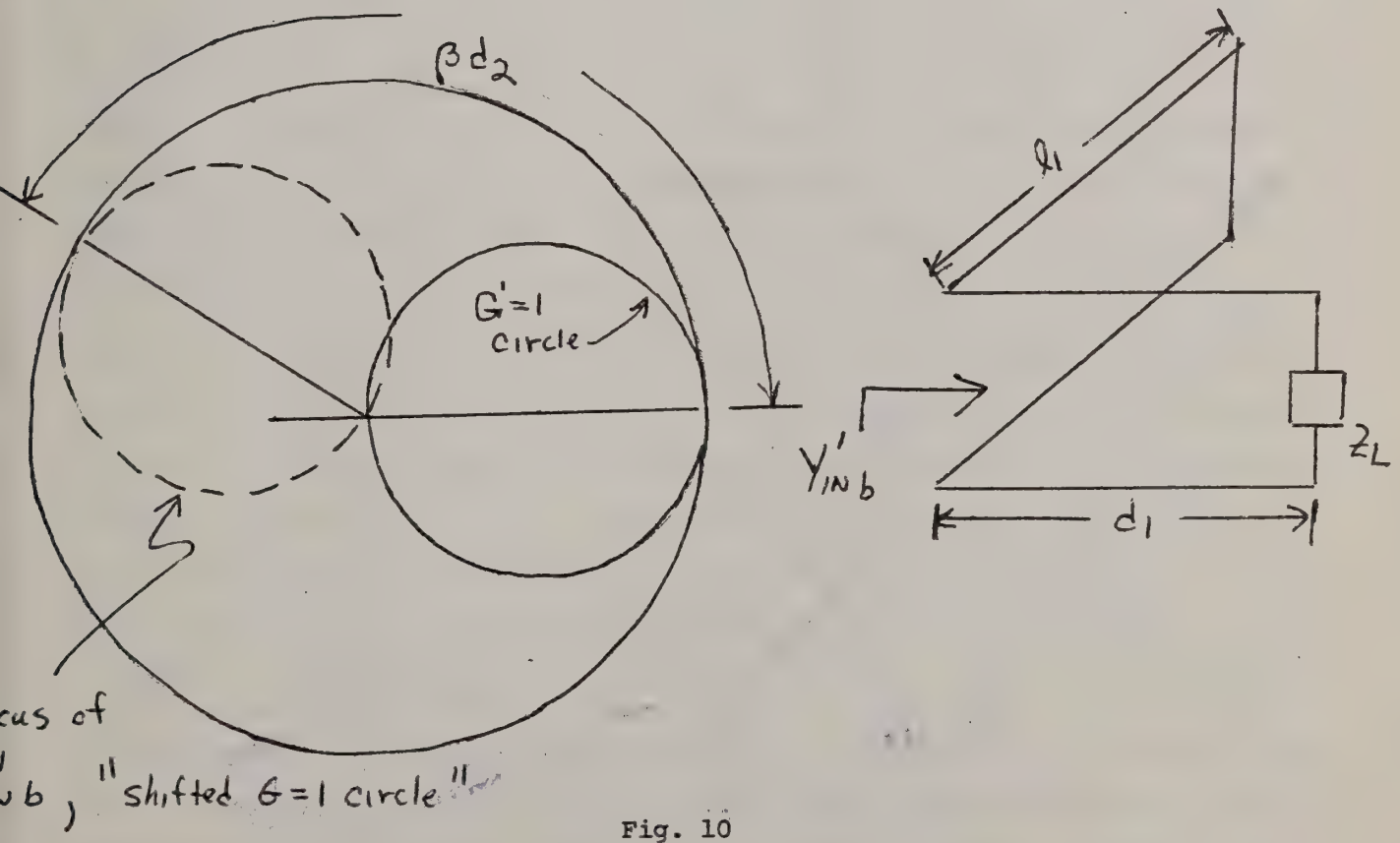



Fig. 10

The next question to answer now is, what is the required length $\ell_1$ of stub number one? To answer this question it is required to find the input admittance of line $d_1$ without the stub one connected, i.e., the input admittance in Fig. 11. One sees here that when the stub is connected it must be true that

$$Y'_{IN_{s_1}} + Y'_{IN_{d_1}} = Y'_{IN_b} \quad . \tag{37}$$

Since all possible $Y'_{IN_b}$ are known (shifted G' = 1 circle), and $Y'_{IN_{d_1}}$ is known when the load and length $d_1$ are given, the stub admittance $Y_{IN_{s_1}}$ is determined.

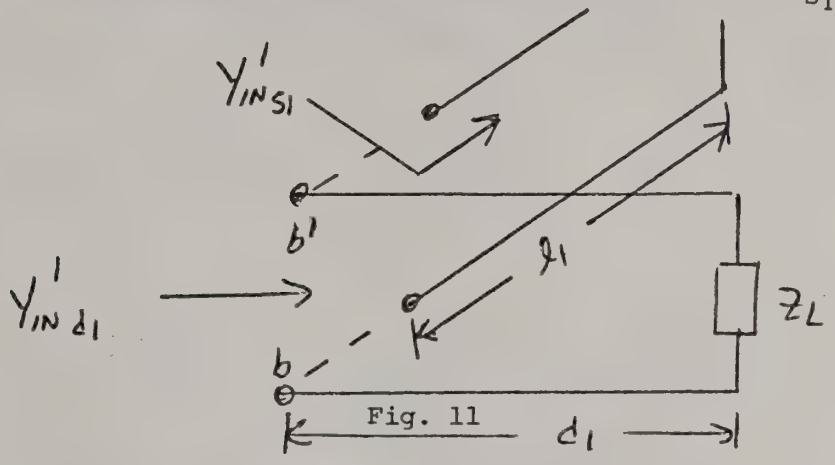



Fig. 11

Because a stub may only present susceptance, it is necessary that the value of G' in the given $Y'_{IN_{d_1}}$ be common to a point on the shifted G' = 1 circle. That is, in Smith chart terms, $Y'_{IN_{d_1}}$ must lie on a conductance circle that intersects the shifted G' = 1 circle. Since in general, not all conductance circles will intersect the shifted G' = 1 circle, not all loads can be matched for a given stub spacing. However, the range of possible matches is still large hence the scheme is a very practical one. From the above considerations the input susceptance of each stub is determined, hence their lengths can be found and the design is complete. These ideas will now be illustrated by way of an example.

Given a load of $Z'_L = 2.0 - j\ 1.0$ and a line length $\beta d_1 = 115$ electrical degrees with a stub spacing $d_2 = \lambda/4$, we proceed to determine the stub lengths required for a match. Figure 12 illustrates the procedure. It is convenient to first find the shifted G' = 1 circle. For this case $\beta d_2 = (2\pi/\lambda)(\lambda/4)$ or 90 electrical degrees (180 polar degrees). This shifted circle is shown in Fig. 12. We next locate the reduced load impedance, convert it to admittance, and transform it through 115 electrical degrees of rotation toward the generator. This is shown in green in Fig. 12 with the result that $Y'_{IN_{d_1}} = 0.8 - j\ 0.85$. The addition of stub number one will now move this point onto the shifted circle. Since the real part is 0.8 we must move along the G' = 0.8 line until intersection with the shifted circle occurs. This intersection occurs at $Y'_{IN_b} = 0.8 - j\ 0.4$, hence the required stub admittance is from (37)

# REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE

MAX. / MIN. IN DECIBELS

MAGNITUDE OF REFLECTION COEFFICIENT

TRANSITION LOSS IN DECIBELS

ATTENUATION SCALE — DECIBELS

$$Y'_{IN_{s_1}} = Y'_{IN_b} - Y'_{IN_{d_1}}$$
$$= 0.8 - j\ 0.4 - 0.8 + j\ 0.85$$
$$= +j\ 0.45 \quad . \tag{38}$$

A short circuited stub with this value of susceptance is found from the chart to be 114 electrical degrees long.

We now move through 90 electrical degrees ($\lambda/4$) toward the generator as dictated by line length $d_2$ to determine that the intersection with the actual $G' = 1$ circle is $Y'_{IN_a} = 1.0 + j\ 0.5$. Hence stub two must provide $-j\ 0.5$ for a match. Again from the chart such a stub would be 63.5 electrical degrees long.

If the intersection with the shifted $G' = 1$ circle had not been obtained it would have been necessary to change either the distance $d_2$ or $d_1$. Again standing waves appear on all sections of line between the load and terminals aa' but the line is matched at aa'.

Balanced to Unbalanced Conversion

The terms "balanced" and "unbalanced" when associated with a transmission line refer to the impedance of each conductor with reference to the ground plane. That is, a line such as 300 ohm television "twin lead" is a balanced line since each conductor with respect to the ground plane presents an equal impedance, while coaxial line is unbalanced since one conductor is common with the ground plane. Obviously if such impedances are not to be discontinuous, balanced and unbalanced lines cannot be connected in tandem even though their characteristic impedances be equal. At low frequencies such an interconnection could be accomplished by insertion of a transformer. In the transformer familiar at low frequencies magnetic coupling predominates. The same transformer operated at high frequencies behaves quite differently since now the previously negligible capacitive coupling becomes significant and will eventually predominate. Hence special techniques must be used at high frequencies to perform balanced to unbalanced coupling.

A. Transmission Line BALUN

BALUN is an acronym for balanced to unbalanced transformer. Figure 13 shows the basic transmission line balun for a 4:1 impedance tranformation. The two quarter wavelength lines are connected in series on the balanced end to yield $Z_0$, while on the unbalanced end the parallel connection yields $Z_0/4$. Note that the normal transmission line modes are excited in each of the $Z_0/2$ lines. Furthermore, when looking into the balun from the right hand side, the ground at the unbalanced line on wire 4 is not seen between terminal 6 and ground because

of the quarter wavelength and thus the balanced line remains balanced to ground. Likewise the ground at terminal 7 is not seen across the unbalanced line at the left, again because of the quarter wavelength of line in between.

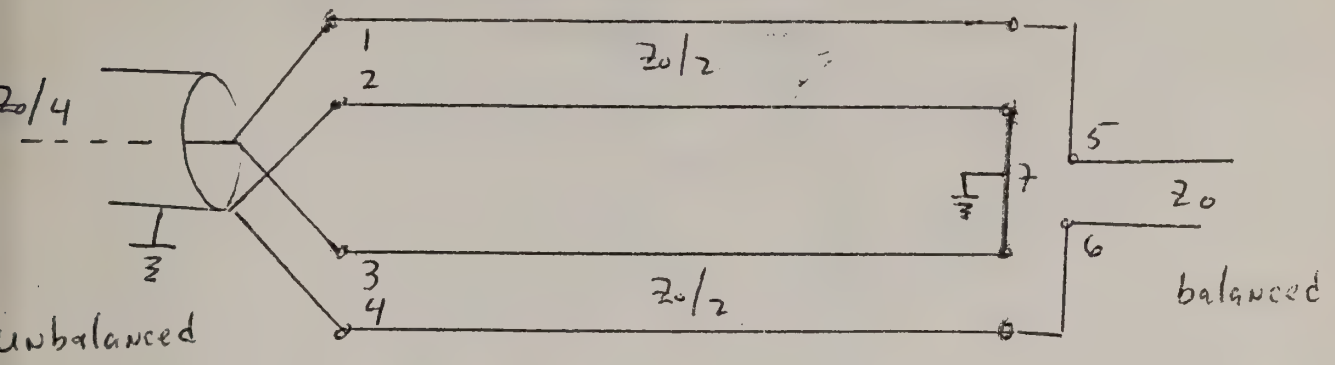



Fig. 13

Such a balun as described above is narrow band because of the required $\lambda/4$ length lines. Broadbanding can be accomplished by winding the $Z_0/2$ lines in the form of a coil. With bifilar winding (maintaining the two wires parallel to each other) the characteristic impedance $Z_0/2$ can be preserved while currents that would exist in the wires because of the grounds on each end are choked because of the coil inductance. Hence coiling the lines yields the same effect as afforded by the quarter wave lines however this method produces a broadband device. Ferrite cores (especially toroidal) are also used for this purpose. The coil balun is an excellent example of a lumped element device that is designed using distributed element concepts in order to extend its range into the high frequency region. Note that in such a device the ever present capacitive coupling is incorporated into the transmission line and hence cannot exert its deleterious effect as a lumped element, furthermore a symmetry is seen here which is not seen in the low frequency transformer.

Many forms of baluns are possible including those for other than the 4:1 impedance ratio. A simple coaxial balum is shown in Fig. 14. Viewing from the balanced line one sees that the impedance to ground from terminal a and from terminal a' is identical because these points are separated by $\lambda/2$ on line 1. Hence the balance on the 300 Ω line is maintained. The impedance transformation however is 4:1 since the unbalanced line terminal voltage and current are V and I respectively. However, the balanced line current is I/2 while the voltage is 2V. That is, the voltage aa' is 2V because the voltage from a' to ground is -V as a result of the $\beta\ell_1 = (2\pi/x)(\lambda/2) = \pi$ radians of phase shift between a and a'.

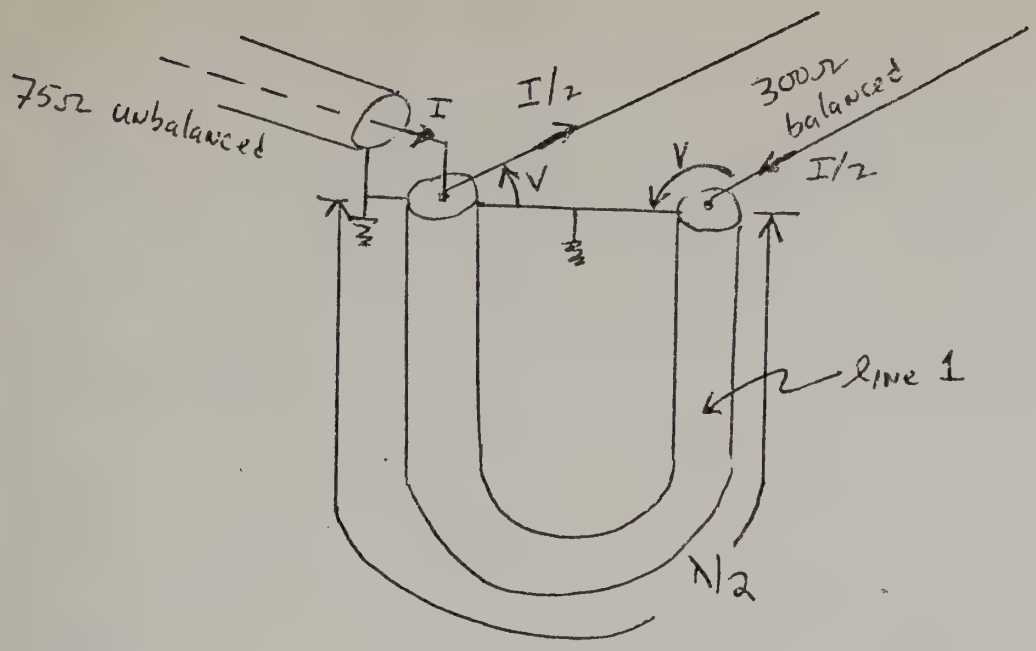



Fig. 14

Figure 15 shows a coaxial balun commonly called a "bazooka." This balun allows a balanced antenna to be fed from an unbalanced line. If the balanced antenna were to be connected directly to the feed line, currents would appear on the outside of the coax sheath which would greatly modify the antenna's characteristics. These currents are prevented and the antenna is maintained balanced by the shorted quarter wave choke which presents an infinite impedance to these currents at the antenna terminals.

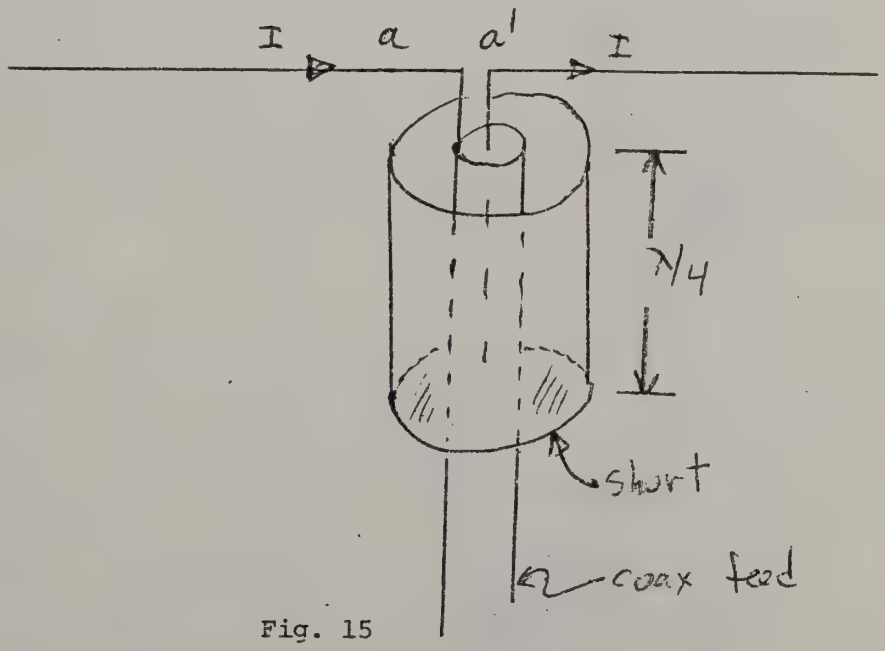



Fig. 15

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